Energy dissipation rate and energy spectrum in high resolution direct numerical simulations of turbulence in a periodic box

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(Received 14 November 2002; accepted 2 December 2002; published 8 January 2003)

High-resolution direct numerical simulations (DNSs) of incompressible homogeneous turbulence in a periodic box with up to 4096^3 grid points were performed on the Earth Simulator computing system. DNS databases, including the present results, suggest that the normalized mean energy dissipation rate per unit mass tends to a constant, independent of the fluid kinematic viscosity ν as $\nu \rightarrow 0$. The DNS results also suggest that the energy spectrum in the inertial subrange almost follows the Kolmogorov $k^{-5/3}$ scaling law, where k is the wavenumber, but the exponent is steeper than $-5/3$ by about 0.1. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1539855]

Direct numerical simulation (DNS) of turbulent flows provides us with detailed turbulence data that are free from experimental ambiguities such as the effects of using Taylor's hypothesis, one dimensional surrogates, etc.[1] However, the degrees of freedom or resolution necessary for DNS increases rapidly with the Reynolds number. The maximum resolution is obviously limited by the available computing memory and speed. To date, DNSs of incompressible turbulence using spectral methods have been limited to a maximum of 1024^3 grid points.

The recently developed Earth Simulator (ES), with a peak performance and main memory of 40 TFlops and 10 TBytes, respectively, provides a new opportunity in this respect. We recently performed a series of DNSs of incompressible turbulence that obeys the Navier–Stokes equations using an alias-free spectral method and up to 4096^3 grid points on the ES. This Letter reports some of the results with a special emphasis on the mean energy dissipation rate $\langle \epsilon \rangle$ per unit mass and the energy spectrum $E(k)$.

The numerical method used for the DNS is similar to that reported in our previous studies.[2,3] In particular, the total energy E was maintained at an almost time-independent constant $(=0.5)$ by introducing negative viscosity in the wavenumber range $k<2.5$. The minimum wavenumber of the DNS was 1. A preliminary report of the DNS with an emphasis on the parallel computing aspect was presented in Ref. 4.

We performed two series of runs, Series 1 and 2, in which the maximum wavenumber $k_{\max}$ and the kinematic viscosity ν were chosen so that $k_{\max}\eta \sim 1$ and 2, respectively, where η is the Kolmogorov length scale defined by $\eta = (\nu^3/\langle \epsilon \rangle)^{1/4}$. In each series, we first executed a low resolution preliminary run (256^3) using an appropriate random initial flow field. The preliminary run was continued until the simulated field reached a statistically quasistationary state, judged by monitoring several one-point statistics. We then reduced the kinematic viscosity ν and started a new DNS with twice the resolution in each direction, using the final state of the lower resolution DNS for the initial conditions. We repeated this process until the resolution reached 4096^3 or 2048^3. The conditions for each run are listed in Table I. We used double precision arithmetic for all of the runs, except Run 4096-1 in which we used single precision arithmetic for the time integration and double precision arithmetic for the convolution sums when evaluating the nonlinear terms in wavevector space. Our preliminary test at a resolution of 1024^3 suggested that the lower arithmetic precision has no significant influence on the energy spectrum. Figure 1 shows the Taylor-scale Reynolds number R_λ versus time t.

DNS data may be used to answer some fundamental questions in turbulence research. Among these is a question about the normalized mean energy dissipation rate $D \equiv \langle \epsilon \rangle L/u'^3$: "Does D tend to a constant independent of the kinematic viscosity ν in the limits as ν tends to zero?" Here, u' is the characteristic velocity of the energy-containing eddies given by $3u'^2 = 2E$ and L is the integral length scale defined by

$$L = \frac{\pi}{2u'^2} \int k^{-1} E(k) dk.$$

The independence of D from ν for large Reynolds number flows is a basic premise in the phenomenology of turbulence; its significance has been emphasized in the literature, as noted by Sreenivasan.[5] He examined the DNS data available

TABLE I. DNS parameters and turbulence characteristics at $t=10$ for Runs 256, 512, 1024, and 2048, and at $t=2.15$ for Run 4096. N is the number of grid points in each direction of the Cartesian coordinates, Δt is the time increment, and λ is the Taylor microlength scale.

		N	R_λ	$k_{\max}$	$\Delta t(\times 10^{-3})$	$\nu(\times 10^{-4})$	$\langle\epsilon\rangle$	L	λ	$\eta(\times 10^{-3})$
Series 1	Run 256-1	256	167	121	1.0	7.0	0.0849	1.13	0.203	7.97
	Run 512-1	512	257	241	1.0	2.8	0.0902	1.02	0.125	3.95
	Run 1024-1	1024	471	483	0.625	1.1	0.0683	1.28	0.0897	2.10
	Run 2048-1	2048	732	965	0.4	0.44	0.0707	1.23	0.0558	1.05
	Run 4096-1	4096	1201	1930	0.25	0.173	0.0668	1.17	0.0360	0.528
Series 2	Run 256-2	256	94	121	1.0	20	0.0936	1.10	0.326	17.1
	Run 512-2	512	173	241	1.0	7.0	0.0795	1.21	0.210	8.10
	Run 1024-2	1024	268	483	0.625	2.8	0.0829	1.12	0.130	4.03
	Run 2048-2	2048	429	965	0.4	1.1	0.0824	1.01	0.0817	2.00

in 1997, when the resolution and the simulated R_λ had reached 512^3 and 250, respectively. Since then, the resolution as well as R_λ have increased substantially. Therefore, it is of interest to revisit this question.

Before analyzing the data, it is important to note that in order to estimate the R_λ-dependence of D, the simulation time must be sufficient to avoid the transient effect. This was stressed by Ishihara and Kaneda[3] (hereafter called IK), who examined the time dependence of L, $\langle\epsilon\rangle$, and D using DNS databases with resolutions of up to 1024^3. Figure 2 shows the time dependence of D in the present Series 1 DNS databases. There is an initial transient period over which D changes rapidly that lasts up to $t=2$. After that period, D is quasistationary (except in Run 256-1 and Run 512-1). Here, since $u'^2=2E/3$ and $E=0.5$, one eddy turnover time $T=L/u'$ is about 2.0 for each run (see Table I).

In the following, we use data at the final time step of each run. Note that due to the size of Run 4096-1, it could only be run up to $t=2.15$. Thus there may be stronger nonstationarity effects on D compared to the other runs.

Figure 3 shows D versus R_λ for the present Series 1 and 2 DNS data. Data from IK and from Sreenivasan,[5] which includes decaying turbulence[6] forced turbulence with negative viscosity[3,6–8] and stochastic forcing,[9,10] are also shown for comparison. The figure suggests that the data form two groups when $R_\lambda<250$, as noted by Sreenivasan.[5] However, the two groups merge together for larger R_λ, and D tends to a constant that is almost independent of R_λ, despite the different run conditions. The constant is approximately 0.4 $\sim$0.5. When $R_\lambda=1201$, the value is ($\approx$0.41) which is in good agreement with recent experiments.[11]

The mean dissipation rate $\langle\epsilon\rangle$ is one of the key variables in various turbulence theories, but it has also been argued that turbulence statistics in the inertial subrange may be better described in terms of the energy flux $\Pi(k)$ rather than $\langle\epsilon\rangle$ that is dominated by dissipation range statistics, where the flux $\Pi(k)$ is defined as $\Pi(k)=-\int_0^k T(k)dk$, and $T(k)$ is the energy transfer function. Spectral closure approximations support this view.

It is widely stated in the literature that in turbulent flows at large Reynolds numbers, (i) there is a wide enough inertial subrange in which $\Pi(k)$ is constant and independent of k, (ii) $\Pi(k)$ is almost stationary, and (iii) $\Pi(k)=\langle\epsilon\rangle$. Many studies and analyses using DNS data have been based implicitly or explicitly on these assumptions. However, as shown by IK, in DNSs with a resolution of up to 1024^3, (i) the range over which the curve $\Pi(k)$ is flat is quite narrow, (ii) the nonstationarity is substantial, and (iii) the difference of $\Pi(k)$ from $\langle\epsilon\rangle$ is also substantial. It would not be surprising if the "inertial range" in a DNS exhibiting (i), (ii), or (iii) is under substantial influence from the nonuniversal statistics of turbulence, such as the initial conditions and low-wavenumber statistics.

It is therefore interesting to examine to what extent (i)–(iii) are satisfied in a DNS with a higher resolution, such as

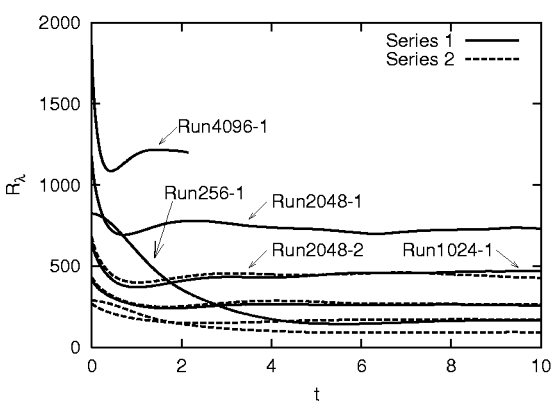

FIG. 1. Taylor microscale Reynolds number R_λ versus time.

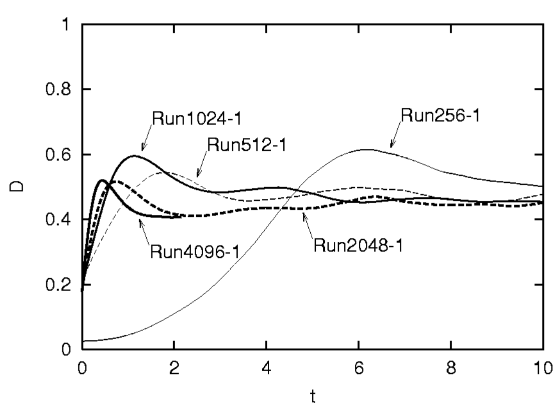

FIG. 2. Time dependence of the normalized mean energy dissipation rate $D=\langle\epsilon\rangle L/u'^3$ in the Series 1 DNSs.

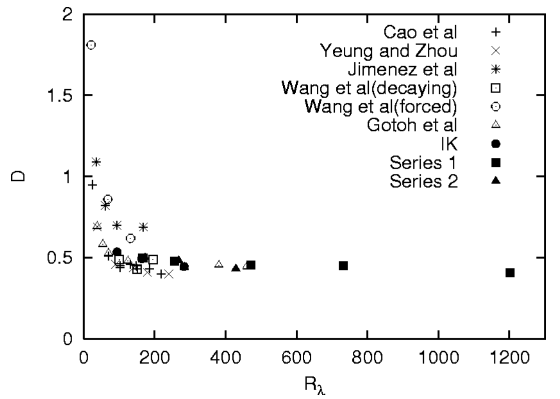

FIG. 3. Normalized energy dissipation rate D versus R_λ from Ref. 5 (data up to $R_\lambda=250$), Ref. 3 (△,●), and the present DNS databases (■,▲).

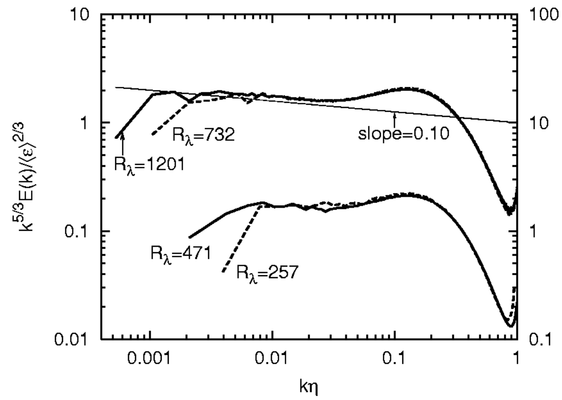

FIG. 5. Compensated energy spectra from DNSs with (A) 512^3, 1024^3, and (B) 2048^3, 4096^3 grid points. Scales on the right and left are for (A) and (B), respectively.

Run 2048-1. Figure 4 shows $\Pi(k)$ at various times in Run 2048-1. The range over which $\Pi(k)$ is nearly constant is quite wide; it is wider than the flat range of the corresponding compensated-energy-spectrum (see Fig. 5). The stationarity is also much better than that of lower resolution DNSs (figures omitted), and $\Pi(k)/\langle\epsilon\rangle$ is close to 1. In the study of the universal features of small-scale statistics of turbulence, if there are any, it is desirable to simulate or realize an inertial subrange exhibiting (i)–(iii) rather than $\overline{\text{(i)}}$–$\overline{\text{(iii)}}$. The present results suggest that a resolution at the level of Run 2048-1 is required for such a simulation. Such DNSs are expected to provide valuable data for the study of turbulence, and in particular for improving our understanding of possible universality characteristics in the inertial subrange.

These considerations motivate us to revisit another simple but fundamental question of turbulence: ''Does the energy spectrum $E(k)$ in the inertial subrange follow Kolmogorov's $k^{-5/3}$ power law at large Reynolds numbers?'' Figure 5 shows the compensated energy spectrum for the present DNSs (the data were plotted in a slightly different manner in our preliminary report[4]). From the simulations with up to $N=1024$, one might think that the spectrum in the range given by

$$E(k)=K_0\epsilon^{2/3}k^{-5/3} \tag{1}$$

with the Kolmogorov constant $K_0=1.6$–1.7 is in good agreement with experiments and numerical simulations (see, for example, Refs. 1, 3, 9, and 10). However, Fig. 5 also shows that the flat region, i.e., the spectrum as described by (1), of the runs with $N=2048$ and 4096 is not much wider than that of the lower resolution simulations. The higher resolution spectra suggest that the compensated spectrum is not flat, but rather tilted slightly, so that it is described by

$$E(k)\propto\epsilon^{2/3}k^{-5/3-\mu_k}, \tag{2}$$

with $\mu_k\neq 0$.

The detection of such a correction to the Kolmogorov scaling, if it in fact exists, is difficult from low-resolution DNS databases. The least square fitting of the data of the 4096^3 resolution simulation for $(d/d\log k)\log E(k)$ to $(-5/3-\mu_k)\log k+b$ (b is a constant) in the range $0.008<k\eta<0.03$ gives $\mu_k=0.10$. The slope with $\mu_k=0.10$ is shown in Fig. 5.

It may be of interest to observe the scaling of the second order moment of velocity, both in wavenumber and physical space. For this purpose, let us consider the structure function

$$S_2(\mathbf{r})=\langle|\mathbf{v}(\mathbf{x}+\mathbf{r},t)-\mathbf{v}(\mathbf{x},t)|^2\rangle,$$

where S_2 may, in general, be expanded in terms of the spherical harmonics as

$$S_2(\mathbf{r})=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}(r)P_n^m(\cos\theta)e^{im\phi}.$$

Here, $r=|\mathbf{r}|$ and θ,ϕ are the angular variables of $\mathbf{r}$ in spherical polar coordinates, P_n^m is the associated Legendre polynomial of order n,m, and $f_{nm}(=f^*_{n,-m})$ is a function of only r, where the asterisk denotes the complex conjugate. The time argument is omitted. For S_2 satisfying the symmetry $S_2(\mathbf{r})=S_2(-\mathbf{r})$, we have $f_{km}=0$ for any odd integer k. In strictly isotropic turbulence, f_{nm} must be zero not only for odd n, but also for any n and m except $n=m=0$. However, our preliminary analysis of the DNS data suggests that the anisotropy is small but nonzero. In such cases, f_{nm} is also small but nonzero, and S_2 itself may not be a good approximation for $f_0=f_{00}$. To improve the approximation for f_0, one might, for example, take the average of S_2 over $\mathbf{r}/r$

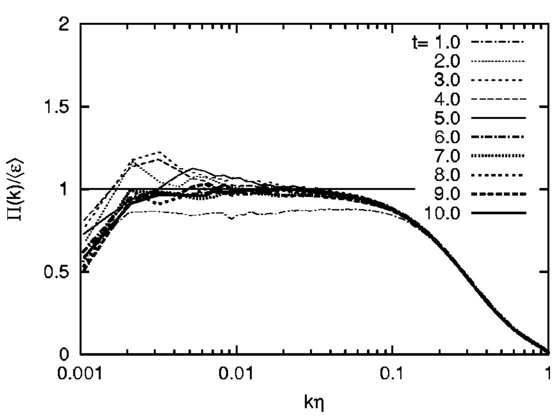

FIG. 4. $\Pi(k)/\langle\epsilon\rangle$ obtained from Run 2048-1.

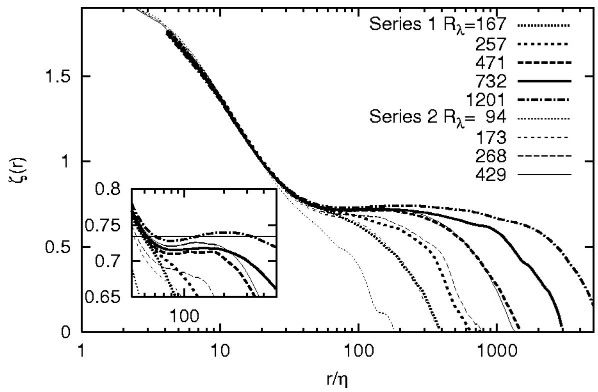

FIG. 6. Local slope $\zeta(r)$ of $f_0(r)$ versus r/η. The inset is an enlargement of the range $40<r/\eta<500$. The straight line shows $\zeta(r)=0.734$.

$=(1,0,0),(0,1,0)$, and $(0,0,1)$. This average is free from f_{2m}; we write it as $[S_2(r)]$. The idea of removing the higher order anisotropy effect is basically similar to that of Arad *et al.*[12]

Figure 6 shows the values of the local slope $\zeta(r) \equiv (d/d\log r)\log[S_2(r)]$. A least square fit of the data of Run 4096-1 to a constant in the range $70<r/\eta<400$, gives $\zeta(r)=0.734\pm0.004$, which implies

$$S_2(\mathbf{r})\propto r^{\zeta_2},\quad \zeta_2=2/3+\mu_r,\quad \mu_r=0.067\pm0.004. \tag{3}$$

The value $\mu_r=0.067$ is a little different from $\mu_k=0.10$ obtained above. Theoretically, we must have $\mu_k=\mu_r$ provided that the inertial subrange is sufficiently wide and $-3<-5/3-\mu_k<-1$. The small difference between the two measured values μ_k and μ_r would not be surprising, if the width of the inertial subrange in the DNS is considered. It would be interesting to improve the statistics by taking time averages, but the simulation time in Run 4096-1 seems too short for taking the time averages.

One might associate the exponents μ_k or μ_r with the intermittency correction. We would, however, prefer to present the data as they are, and leave the interpretation of the exponent to readers or to a future study.

ACKNOWLEDGMENTS

The authors would like to thank Dr. T. Sato, the Director-general of the Earth Simulator Center, for his warm encouragement of this study. The computations were performed on the Earth Simulator. The authors are grateful to Dr. R. Rubinstein for providing stimulating discussion. This work was supported by the "Research for the Future" Program of the Japan Society for the Promotion of Science (JSPS) under Project No. JSPS-RFTF97P01101, a Grant-in-Aid for Scientific Research [Grant No. (B)14340033] from JSPS, and Grants-in-Aid for Exploratory Research (Grant No. 14654073) and for Young Scientists [Grant No. (B)13750059] from the Ministry of Education, Culture, Sports, Science, and Technology, of Japan.

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