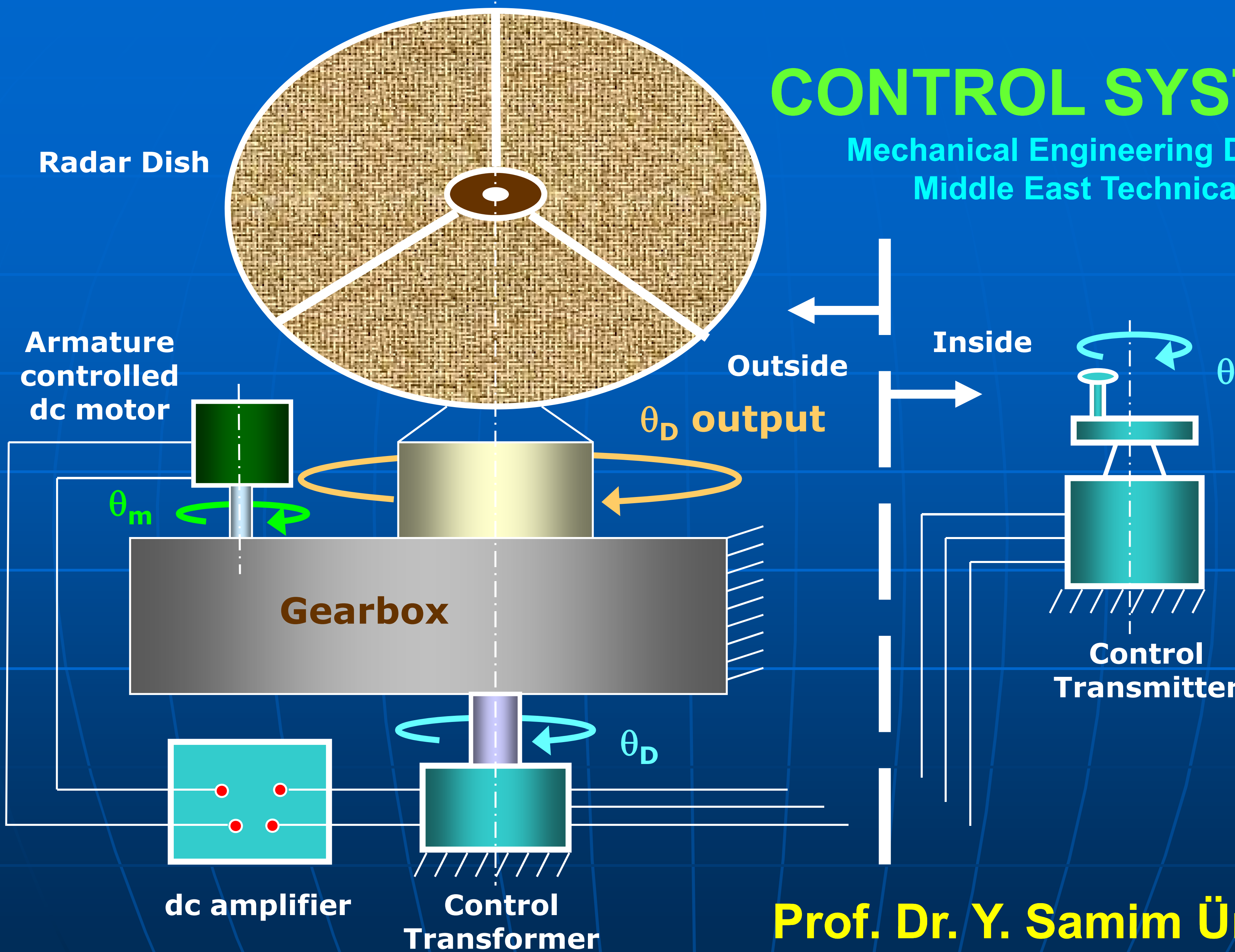
ME 304
CONTROL SYSTEMS

Mechanical Engineering Department,
Middle East Technical University



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CH IX

COURSE OUTLINE

I. INTRODUCTION & BASIC CONCEPTS
II. MODELING DYNAMIC SYSTEMS
III. CONTROL SYSTEM COMPONENTS
IV. STABILITY
V. TRANSIENT RESPONSE
VI. STEADY STATE RESPONSE
VII. DISTURBANCE REJECTION
VIII. BASIC CONTROL ACTIONS & CONTROLLERS
IX. FREQUENCY RESPONSE ANALYSIS
X. SENSITIVITY ANALYSIS
XI. ROOT LOCUS ANALYSIS

FREQUENCY RESPONSE - OBJECTIVES

In this chapter :

- **A short introduction to the steady state response of control systems to sinusoidal inputs will be given.**
- **Frequency domain specifications for a control system will be examined.**
- **Bode plots and their construction using asymptotic approximations will be presented.**

FREQUENCY RESPONSE – INTRODUCTION

Nise Ch. 10

- **In frequency response analysis of control systems, the steady state response of the system to sinusoidal input is of interest.**
- **The frequency response analyses are carried out in the frequency domain, rather than the time domain.**
- **It is to be noted that, time domain properties of a control system can be predicted from its frequency domain characteristics.**

FREQUENCY RESPONSE - INTRODUCTION

- **For an LTI system the Laplace transforms of the input and output are related to each other by the transfer function, T(s).**

- **In the frequency response analysis, the system is excited by a sinusoidal input of fixed amplitude and varying frequency.**

FREQUENCY RESPONSE - INTRODUCTION

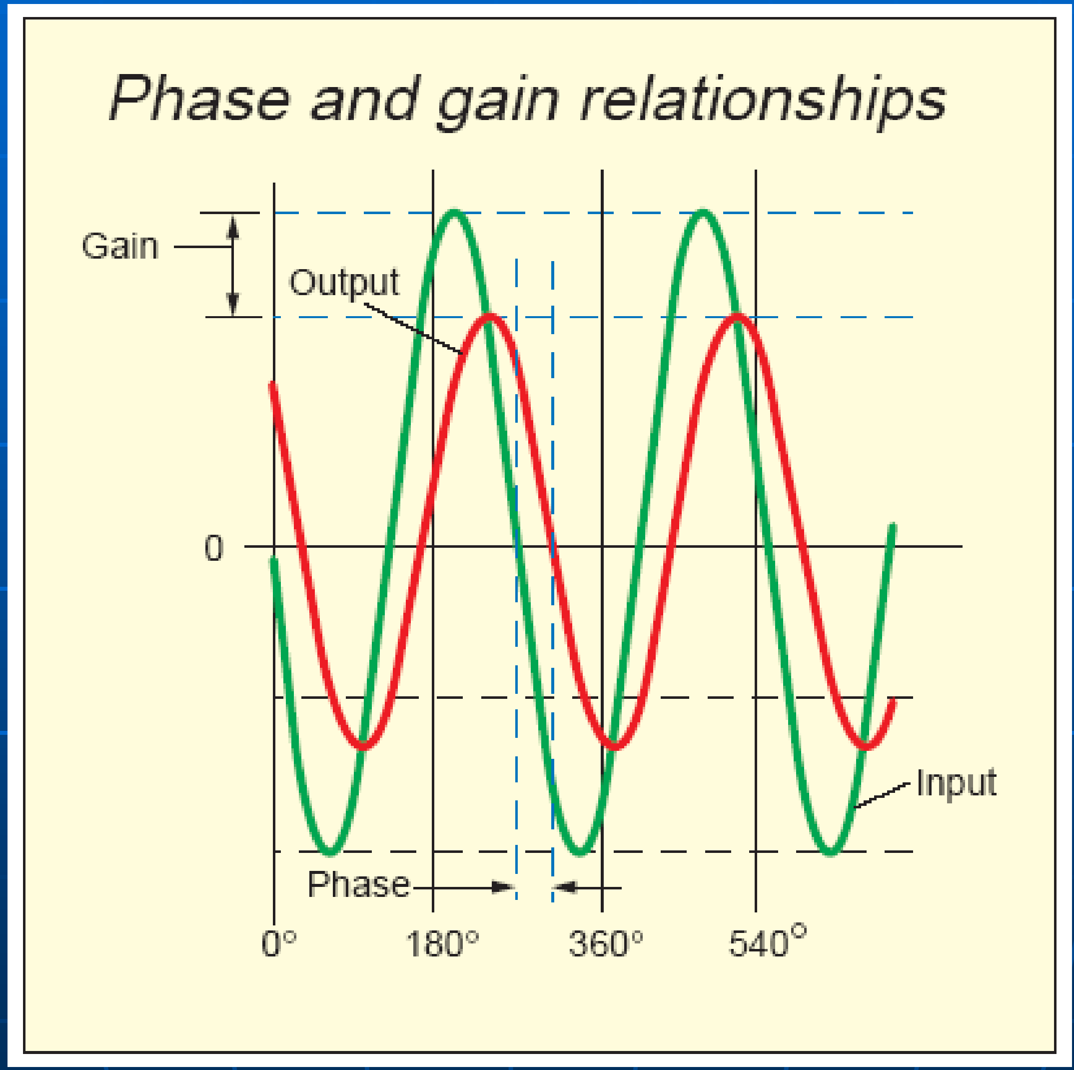
- **Let us subject a stable LTI system to a sinusoidal input of amplitude R and frequency ω in time domain.**

$$r(t)=R\sin(\omega t)$$

- **The steady state output of the system will be again a sinusoidal signal of the same frequency, but probably with a different amplitude and phase.**

$$c(t)=C\sin(\omega t+\phi)$$

FREQUENCY RESPONSE - INTRODUCTION



FREQUENCY RESPONSE - INTRODUCTION

- To carry out the same process in the frequency domain for sinusoidal steady state analysis, one replaces the Laplace variable **s** with

  $$s=j\omega$$

  in the input output relation

  $$C(s)=T(s)R(s)$$

  with the result

  $$C(j\omega)=T(j\omega)R(j\omega)$$

FREQUENCY RESPONSE - INTRODUCTION

- **The input, output, and the transfer function have now become complex and thus they can be represented by their magnitudes and phases.**

- **Input :** $R(j\omega) = |R(j\omega)| \angle R(j\omega)$

- **Output :** $C(j\omega) = |C(j\omega)| \angle C(j\omega)$

- **Transfer Function :** $T(j\omega) = |T(j\omega)| \angle T(j\omega)$

FREQUENCY RESPONSE - INTRODUCTION

- **With similar expressions for the input and the transfer function, the input output relation in the frequency domain consists of the magnitude and phase expressions :**

$$C(j\omega)=T(j\omega)R(j\omega)$$

$$|C(j\omega)| = |T(j\omega)||R(j\omega)|$$

$$\angle C(j\omega) = \angle T(j\omega) + \angle R(j\omega)$$

FREQUENCY RESPONSE - INTRODUCTION

- **For the input and output described by**

$$r(t)=R\sin(\omega t)$$

$$c(t)=C\sin(\omega t+\phi)$$

the amplitude and the phase of the output can now be written as

$$C = R\left|T(j\omega)\right|$$

$$\phi = \angle T(j\omega)$$

FREQUENCY RESPONSE

- **Consider the transfer function for the general closed loop system.**

$$T(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$$

For the steady state behaviour, insert $s=j\omega$.

$$T(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

$T(j\omega)$ is called the Frequency Response Function (FRF) or Sinusoidal Transfer Function.

FREQUENCY RESPONSE

- **The frequency response function can be written in terms of its magnitude and phase.**

$$\mathbf{T(j\omega) = \left|T(j\omega)\right| \angle T(j\omega)}$$

Since this function is complex, it can also be written in terms of its real and imaginary parts.

$$\mathbf{T(j\omega) = Re\left[T(j\omega)\right] + jIm\left[T(j\omega)\right]}$$

FREQUENCY RESPONSE

- Remember that for a complex number be expressed in its real and imaginary parts :

$$z = a + bj$$

- the magnitude is given by :

$$|z| = \sqrt{(a+bj)(a-bj)} = \sqrt{a^2 + b^2}$$

- the phase is given by :

$$\angle z = \tan^{-1}\frac{b}{a}$$

FREQUENCY RESPONSE

- **The magnitude and phase of the frequency response function are given by :**

$$\left|T(j\omega)\right| = \left|\frac{G(j\omega)}{1+G(j\omega)H(j\omega)}\right| = \frac{\left|G(j\omega)\right|}{\left|1+G(j\omega)H(j\omega)\right|}$$

$$\angle T(j\omega) = \angle G(j\omega) - \angle\left[1+G(j\omega)H(j\omega)\right]$$

These are called the gain and phase characteristics.

FREQUENCY RESPONSE – Example 1a

- **For a system described by the differential equation**

$$\ddot{x} + 2\dot{x} = y(t)$$

determine the steady state response $x_{ss}(t)$ for a pure sine wave input

$$y(t) = 3\sin(0.5t)$$

FREQUENCY RESPONSE – Example 1b

- **The transfer function is given by**

$$\ddot{x}+2\dot{x}=y(t) \quad \Rightarrow \quad T(s)=\frac{X(s)}{Y(s)}=\frac{1}{s(s+2)}$$

Insert s=jω to get :

$$T(j\omega)=\frac{1}{j\omega(j\omega+2)}$$

For ω=0.5 [rad/s]:

$$T(0.5j)=\frac{1}{0.5j(0.5j+2)}=\frac{1}{-0.25+j}$$

FREQUENCY RESPONSE – Example 1c

- **Multiply and divide by the complex conjugate.**

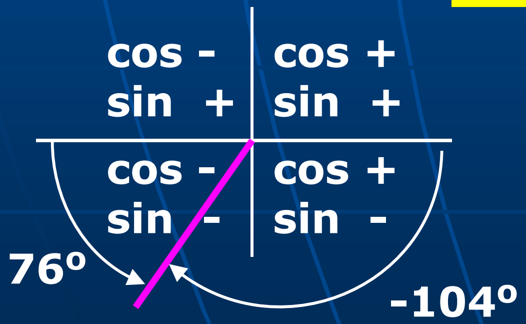
$$T(0.5j)=\left(\frac{1}{-0.25+j}\right)\left(\frac{-0.25-j}{-0.25-j}\right)=\frac{-0.25-j}{1+0.0625}$$

$$T(0.5j)=-0.235-0.941j$$

- **Determine the magnitude and the angle.**

$$\left|T(0.5j)\right|=\sqrt{(-0.235)^2+(-0.941)^2}=0.97$$

$$\angle T(0.5j)=\tan^{-1}\frac{-0.941}{-0.235}=-104^{o}$$



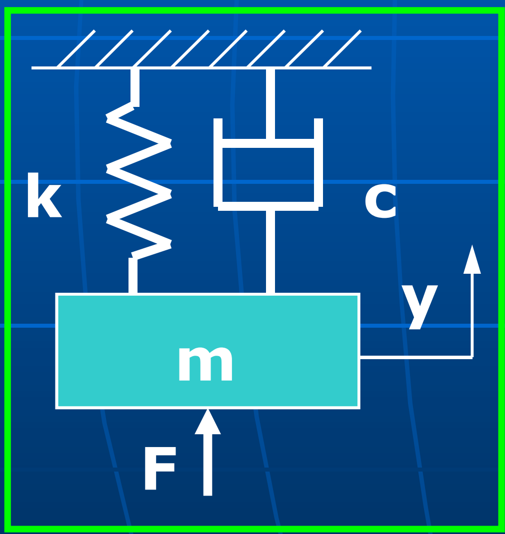
FREQUENCY RESPONSE – Example 1d

- **The steady state response is then given by :**

$$x_{ss}(t) = 3(0.97)\sin\left(0.5t - 104^{o}\right)$$
$$= 2.91\sin\left(0.5t - 104^{o}\right)$$

FREQUENCY RESPONSE – Example 2a

- **Express the transfer function (input : F, output : y) in terms of its magnitude and phase.**



$$m\ddot{y} + c\dot{y} + ky = F$$

$$G(s) = \frac{1}{ms^2 + cs + k}$$

FREQUENCY RESPONSE – Example 2b

- **Insert s=jω in the transfer function to obtain the frequency response function.**

$$G(s)=\frac{1}{ms^2+cs+k}$$

$$T(j\omega)=\frac{1}{m(\omega j)^2+c(\omega j)+k}=\frac{1}{\left(k-m\omega^2\right)+c\omega j}$$

- **Write the FRF in a+bj form.**

FREQUENCY RESPONSE – Example 2c

- **Multiply and divide the FRF expression with the complex conjugate of its denominator.**

$$T(j\omega)=\frac{1}{\left(k-m\omega^2\right)+c\omega j}\,\frac{\left(k-m\omega^2\right)-c\omega j}{\left(k-m\omega^2\right)-c\omega j}=\frac{\left(k-m\omega^2\right)-c\omega j}{\left(k-m\omega^2\right)^2+\left(c\omega\right)^2}$$

$$T(j\omega)=\left[\frac{\left(k-m\omega^2\right)}{\left(k-m\omega^2\right)^2+\left(c\omega\right)^2}\right]+\left[\frac{-c\omega}{\left(k-m\omega^2\right)^2+\left(c\omega\right)^2}\right]j$$

$$T(j\omega)=\mathrm{Re}\left[T(j\omega)\right]+\mathrm{Im}\left[T(j\omega)\right]j$$

FREQUENCY RESPONSE – Example 2d

- **Obtain the magnitude and phase of the frequency response function.**

$$\left|z\right| = \sqrt{a^2 + b^2}$$

$$\left|T(j\omega)\right| = \sqrt{\frac{\left(k - m\omega^2\right)^2 + (c\omega)^2}{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]^2}} = \frac{1}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}$$

$$\angle z = \tan^{-1}\frac{b}{a} \quad \Rightarrow \quad \angle T(j\omega) = \tan^{-1}\frac{-c\omega}{\left(k - m\omega^2\right)}$$

FREQUENCY RESPONSE – Example 3a

- **The open loop transfer function of a control system is given as :**

$$G(s) = \frac{300(s+100)}{s(s+10)(s+40)}$$

- **Determine an expression for the phase angle of G(jw) <u>in terms of</u> the angles of its basic factors. Calculate its value at a frequency of 28.3 rad/s.**
- **Determine the expression for the magnitude of G(jw) <u>in terms of</u> the magnitudes of its basic factors . Find its value in dB at a frequency of 28.3 rad/s.**

$$G(s)=\frac{300(s+100)}{s(s+10)(s+40)}$$

FREQUENCY RESPONSE – Example 3b

$$\angle G(j\omega) = \angle 300 + \angle G(j\omega+100) - \angle G(j\omega) - \angle G(j\omega+10) - \angle G(j\omega+40)$$

$$= 0 + \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{40}\right)$$

$$= 0^{o} + \tan^{-1}\left(\frac{\omega}{100}\right) - 90^{o} - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{40}\right)$$

$$\angle G(28.3j) = 0^{o} + \tan^{-1}\left(\frac{28.3}{100}\right) - 90^{o} - \tan^{-1}\left(\frac{28.3}{10}\right) - \tan^{-1}\left(\frac{28.3}{40}\right)$$

$$= 0^{o} + 15.8^{o} - 90^{o} - 70.5^{o} - 35.3^{o} = -180^{o}$$

$$G(s)=\frac{300(s+100)}{s(s+10)(s+40)}$$

FREQUENCY RESPONSE – Example 3c

$$|G(j\omega)|=\frac{|300||j\omega+100|}{|j\omega||j\omega+10||j\omega+40|}$$

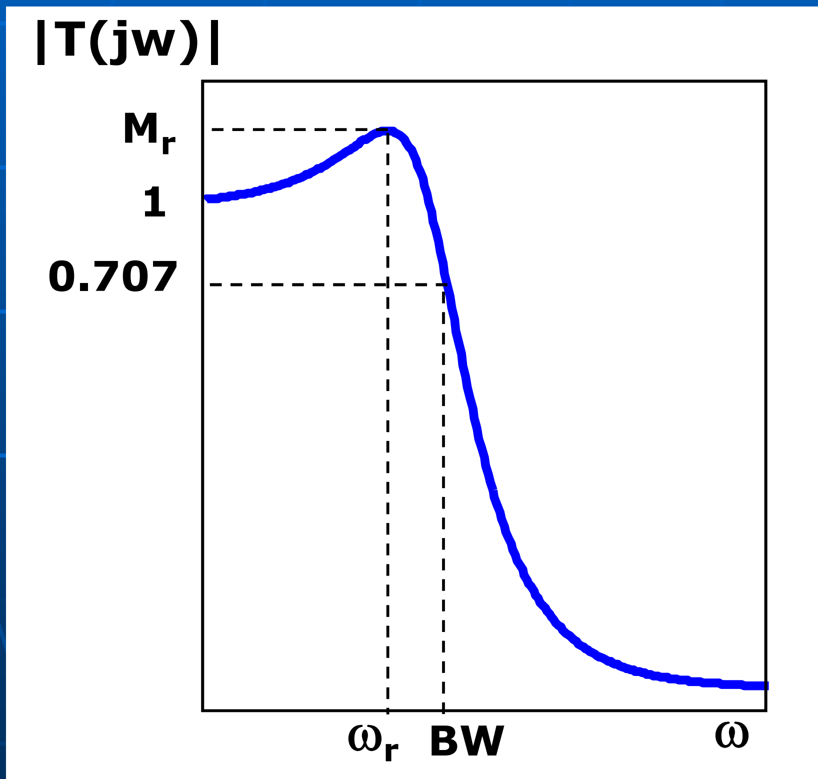
$$=\frac{300\sqrt{\omega^2+100^2}}{\omega\sqrt{\omega^2+10^2}\sqrt{\omega^2+40^2}}$$

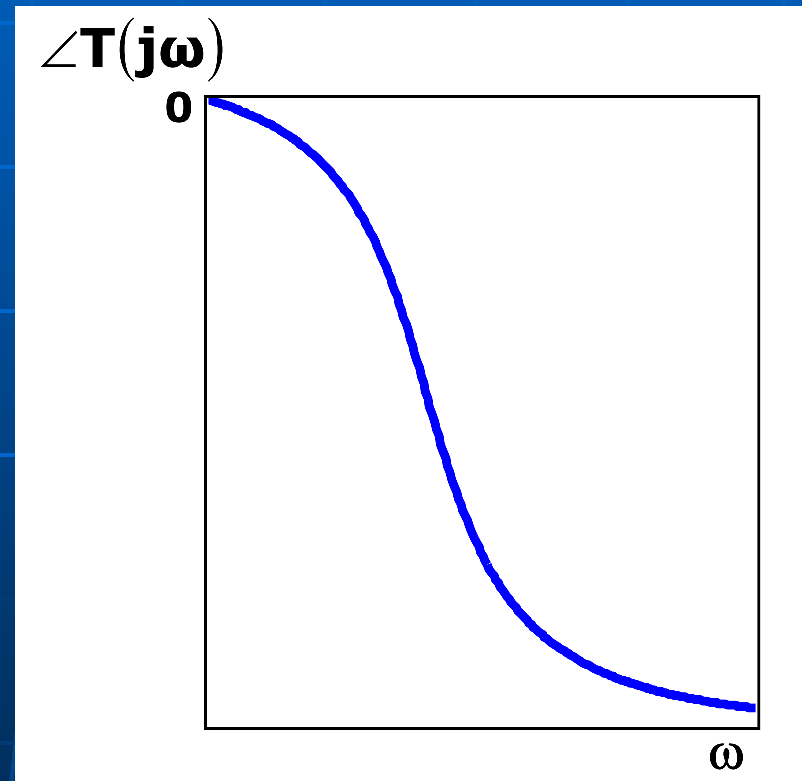
$$|G(28.3j)|=\frac{300\sqrt{28.3^2+100^2}}{28.3\sqrt{28.3^2+10^2}\sqrt{28.3^2+40^2}}$$

$$=\frac{(300)(103.9)}{(28.3)(30.0)(49.0)}=0.749$$

FREQUENCY RESPONSE

- Typical gain and phase characteristics of a **closed loop** system.





FREQUENCY DOMAIN SPECIFICATIONS

- **Similar to transient response specifications in time domain, frequency response specifications are defined.**

  - Resonant peak, M_r,

  - Resonant frequency, ω_r,

  - Bandwidth, BW,

  - Cutoff Rate.

FREQUENCY DOMAIN SPECIFICATIONS

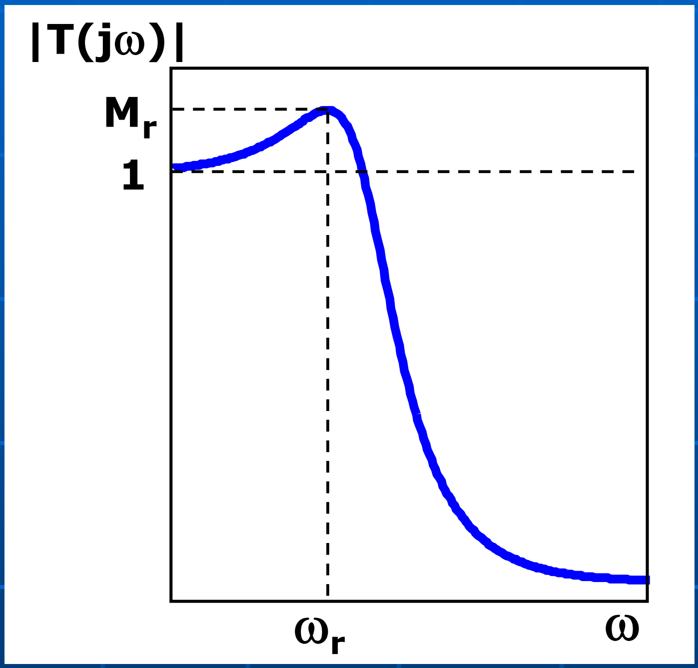
- **Resonant peak, M_r :**

  This is the maximum value of the transfer function magnitude $|T(j\omega)|$.

  M_r depends on the damping ratio ξ only and indicates the relative stability of a stable closed loop system.

  A large M_r results in a large overshoot of the step response.

  As a rule of thumb, M_r should be between 1.1 and 1.5.



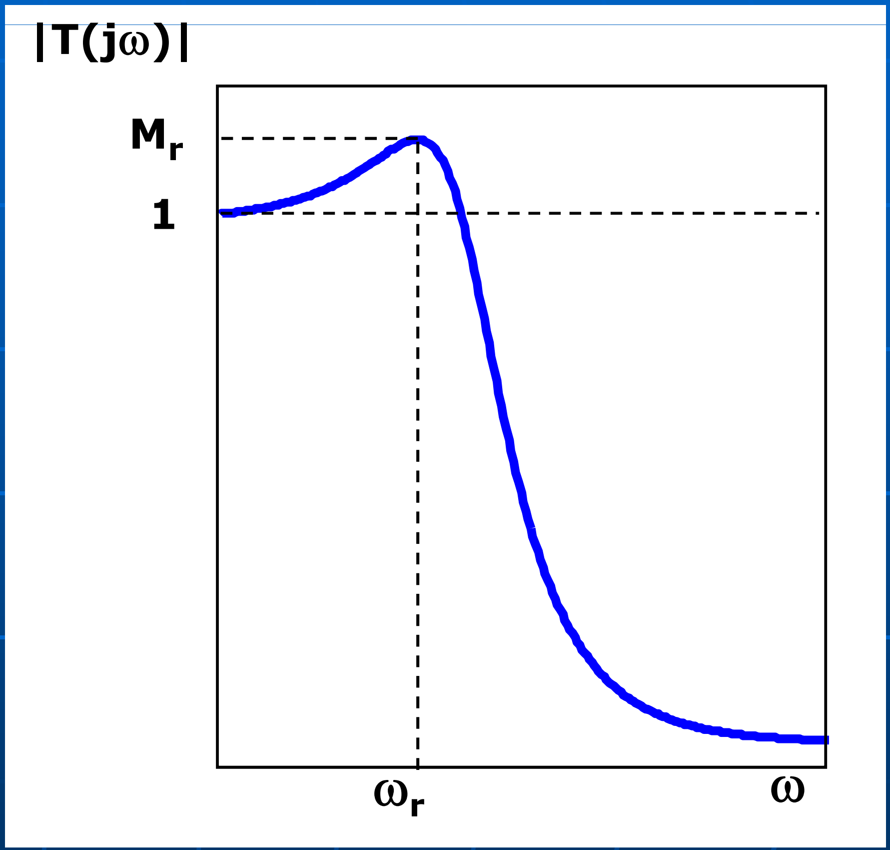
$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

FREQUENCY DOMAIN SPECIFICATIONS

- **Resonant frequency, ω_r :**

  This is the frequency at which the resonant peak is obtained.

$$\omega_r = \omega_n \sqrt{1 - 2\xi^2}$$

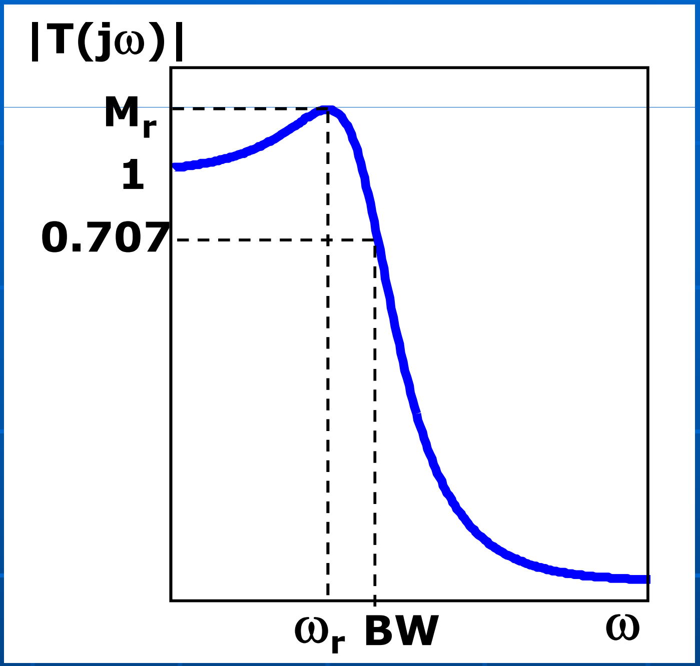


Note that resonant frequency is different than both the undamped and damped natural frequencies!

FREQUENCY DOMAIN SPECIFICATIONS

- **Bandwidth, BW :**

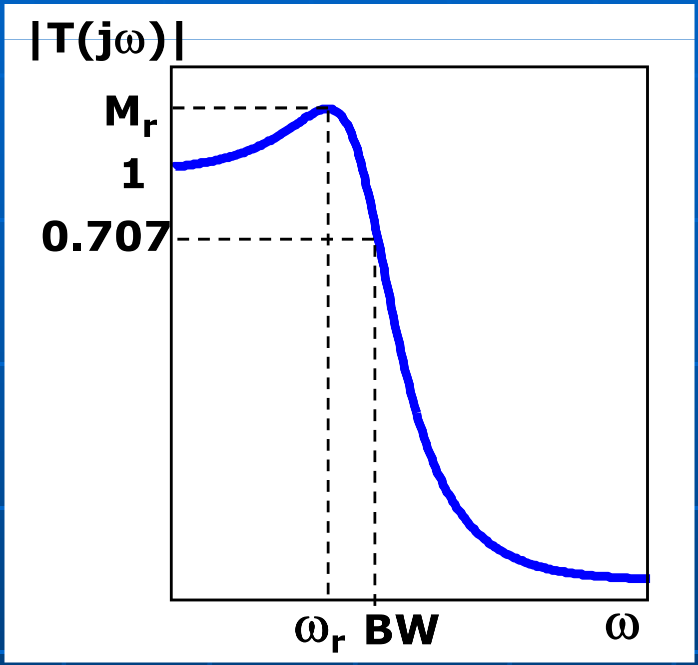
  This is the frequency at which the magnitude of the frequency response function, $|T(j\omega)|$, drops to 0.707 of its zero frequency value.



- **BW is directly proportional to ω_n and gives an indication of the transient response characteristics of a control system. The larger the bandwidth is, the faster the system responds.**

FREQUENCY DOMAIN SPECIFICATIONS

- **Bandwidth, BW :**
- **It is also an indicator of robustness and noise filtering characteristics of a control system.**

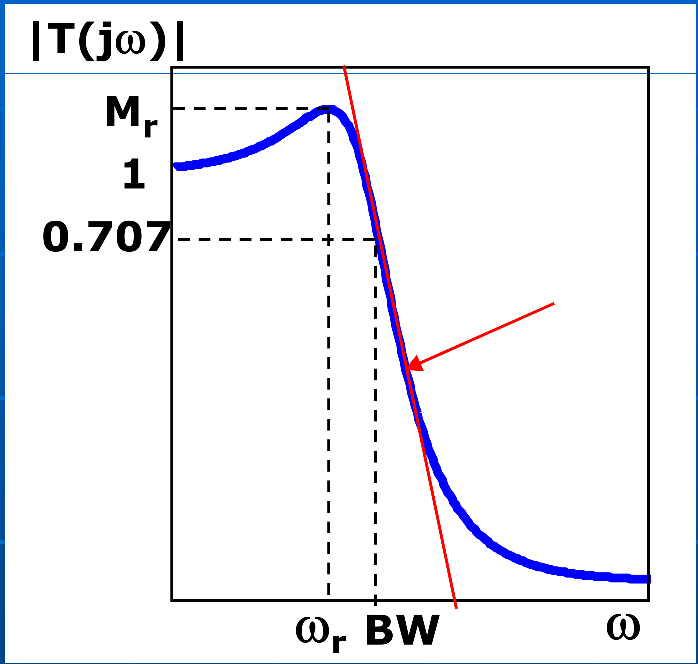


$$\omega_{BW} = \omega_n \sqrt{\left(1 - 2\xi^2\right) + \sqrt{4\xi^4 - 4\xi^2 + 2}}$$

FREQUENCY DOMAIN SPECIFICATIONS

- **Cut-off Rate :**

  This is the slope of the magnitude of the frequency response function, |T(jω)|, at higher (above resonant) frequencies.



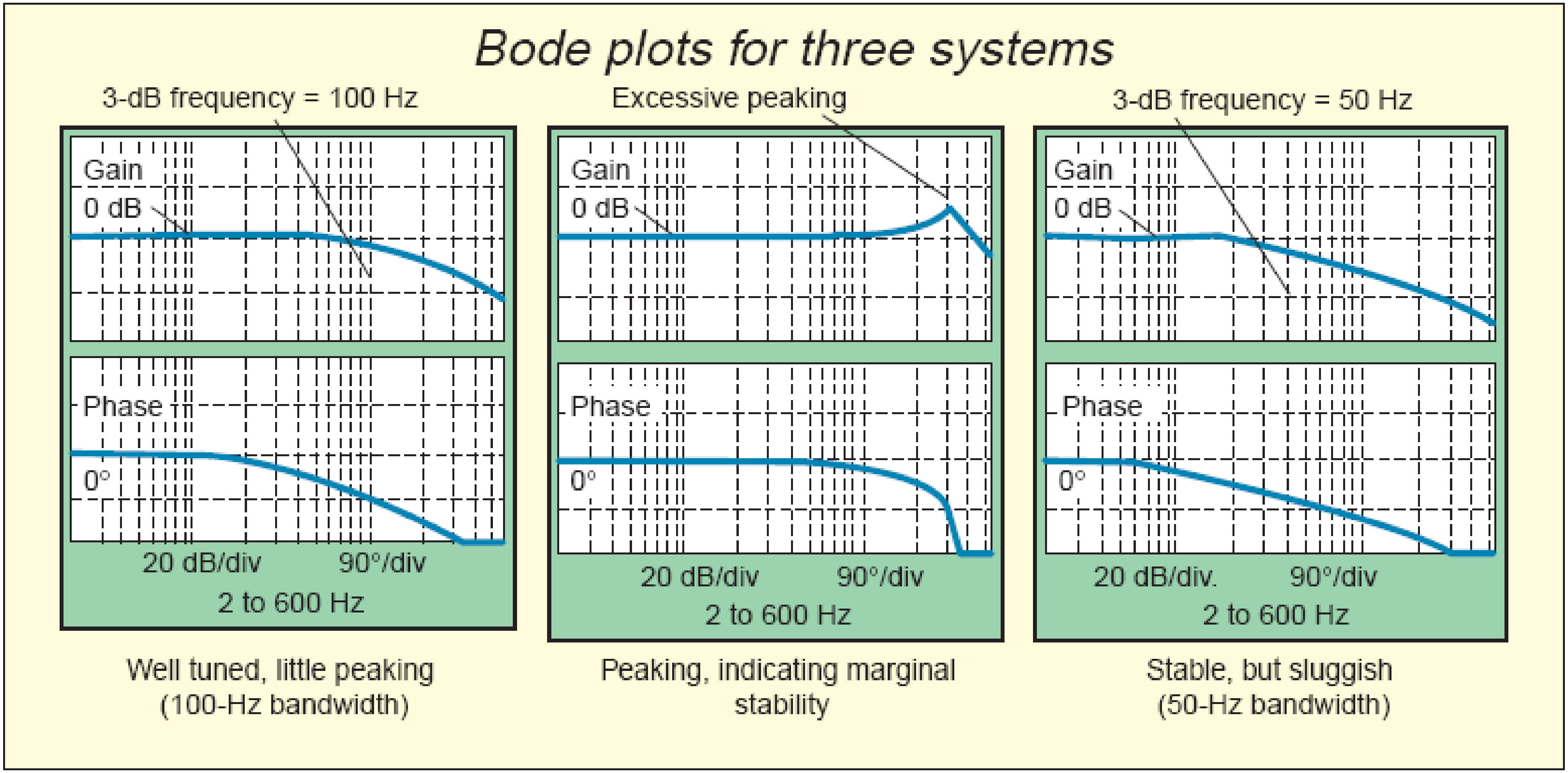
- **It indicates the ability of a system to distinguish signals from noise.**
- **Two systems having the same bandwidth can have different cutoff rates.**

BODE PLOT

Dorf & Bishop Ch. 8, Ogata Ch. 8

- **The Bode plot** of a transfer function is a useful graphical tool for the analysis and design of linear control systems in the frequency domain.
- The Bode plot has the advantages that
  - it can be sketched approximately using straightline segments without using a computer.
  - relative stability characteristics are easily determined, and
  - effects of adding controllers and their parameters are easily visualized.

BODE PLOT



Bode plots for three systems
3-dB frequency = 100 Hz
Excessive peaking
3-dB frequency = 50 Hz
Gain
0 dB
Phase
0°
20 dB/div
90°/div
2 to 600 Hz
Gain
0 dB
Phase
0°
20 dB/div
90°/div
2 to 600 Hz
Gain
0 dB
Phase
0°
20 dB/div.
90°/div
2 to 600 Hz
Well tuned, little peaking (100-Hz bandwidth)
Peaking, indicating marginal stability
Stable, but sluggish (50-Hz bandwidth)

BODE PLOT

Nise Section 10.2

- **The Bode plot consists of two plots drawn on semi-logarithmic paper.**

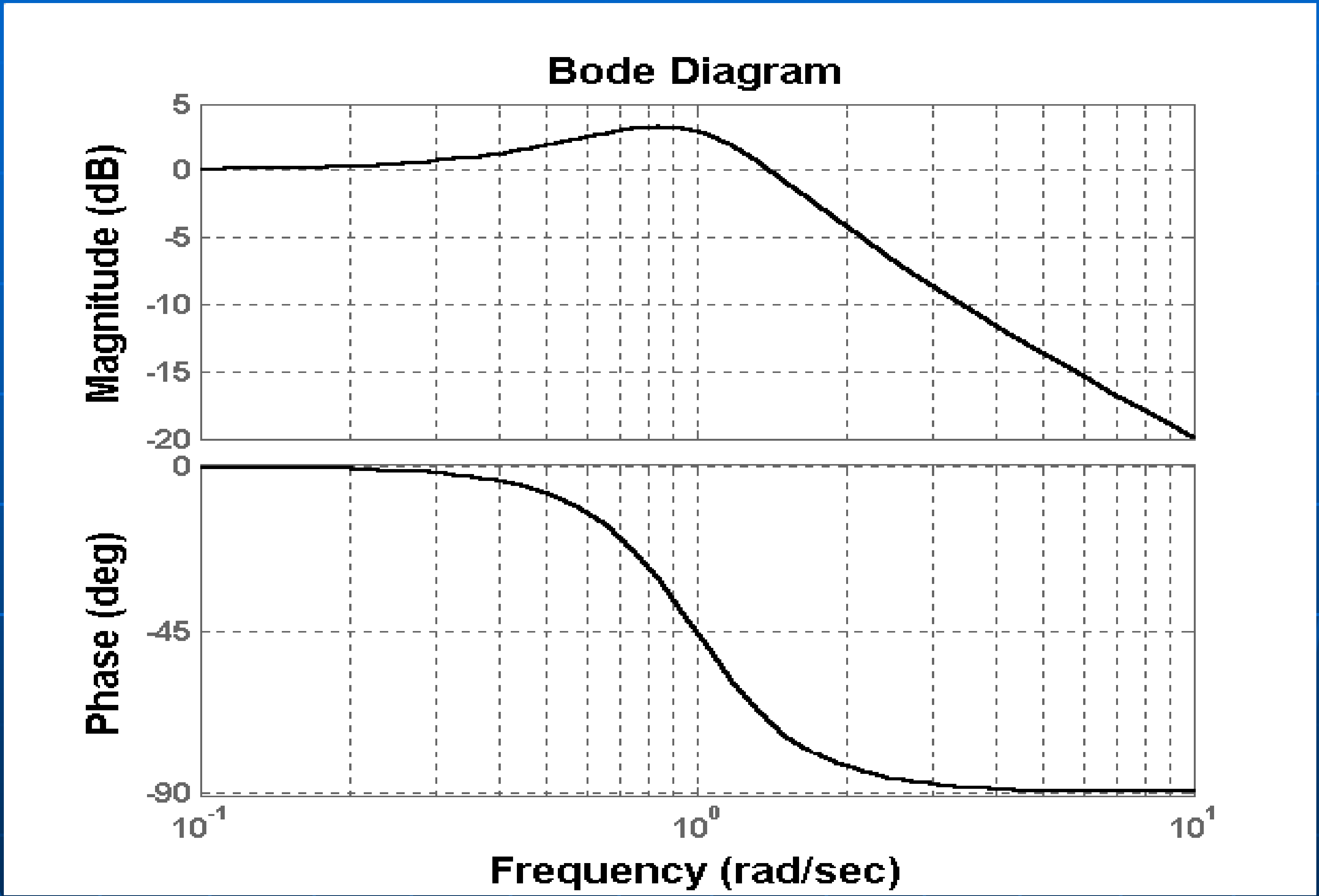
  1. Magnitude of the frequency response function in decibels, i.e.,

  $$20 \log|T(j\omega)|$$

  on a linear scale versus frequency on a logarithmic scale.

  2. Phase of the frequency response function on a linear scale versus frequency on a logarithmic scale.

BODE PLOT



Bode Diagram
Magnitude (dB)
5
0
-5
-10
-15
-20
Phase (deg)
0
-45
-90
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

BODE PLOT

- **It is possible to construct the Bode plots of the open loop transfer functions, but the closed loop frequency response is not so easy to plot.**
- **It is also possible, however, to obtain the closed loop frequency response from the open loop frequency response.**
- **Thus, it is usual to draw the Bode plots of the open loop transfer functions. Then the closed loop frequency response can be evaluated from the open loop Bode plots.**

BODE PLOT

- It is possible to construct the Bode plots by adding the contributions of the basic factors of T(jω) by graphical addition.
- Consider the following general transfer function.

$$T(s)=\frac{K\prod_{p=1}^{P}\left(1+T_p s\right)}{s^N\prod_{m=1}^{M}\left(1+\tau_m s\right)\prod_{q=1}^{Q}\left(1+2\xi_q\frac{s}{\omega_{n_q}}+\frac{s^2}{\omega_{n_q}^2}\right)}$$

$$T(s)=\frac{K\prod_{p=1}^{P}\left(1+T_p s\right)}{s^N\prod_{m=1}^{M}\left(1+\tau_m s\right)\prod_{q=1}^{Q}\left(1+2\xi_q\frac{s}{\omega_{n_q}}+\frac{s^2}{\omega_{n_q}^2}\right)}$$

BODE PLOT

- The logarithmic magnitude of T(jω) can be obtained by summation of the logarithmic magnitudes of individual terms.

$$\log\left|T(j\omega)\right|=\log K+\sum_{p}^{P}\log\left|1+j\omega\tau_p\right|-$$

$$-\log\left|(j\omega)^N\right|-\sum_{m}^{M}\log\left|1+j\omega\tau_m\right|-\sum_{q}^{Q}\log\left|1+\frac{2\xi_q}{\omega_{n_q}}j\omega+\left(\frac{j\omega}{\omega_{n_q}}\right)^2\right|$$

$$T(s)=\frac{K\prod_{p=1}^{P}\left(1+T_p s\right)}{s^N\prod_{m=1}^{M}\left(1+\tau_m s\right)\prod_{q=1}^{Q}\left(1+2\xi_q\frac{s}{\omega_{n_q}}+\frac{s^2}{\omega_{n_q}^2}\right)}$$

BODE PLOT

- **Similarly, the phase of T(jω) can be obtained by simple summation of the phases of individual terms.**

$$\phi=\angle\left|T(j\omega)\right|=\sum_{p}^{P}\tan^{-1}\omega\tau_p-N\left(90^o\right)-\sum_{m}^{M}\tan^{-1}\omega\tau_m-\sum_{q}^{Q}\tan^{-1}\left(\frac{2\xi_q\omega_{n_q}\omega}{\omega_{n_q}^2-\omega^2}\right)$$

BODE PLOT

- **Therefore, any transfer function can be constructed from the four basic factors :**

  1. Gain, K - a constant,

  2. Integral, $1/j\omega$, or derivative factor, $j\omega$ – pole or zero at the origin,

  3. First order factor – simple lag, $1/(1+j\omega T)$, or lead $1+j\omega T$ (real pole or zero),

  4. Quadratic factor – quadratic lag or lead.

$$1\Bigg/\left[1+2\xi\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right] \quad \text{or} \quad \left[1+2\xi\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right]$$

BODE PLOT

Some useful definitions :

- **The magnitude is normally specified in decibels [dB].**

  The value of M in decibels is given by :

  $$M[dB]=20\log M$$

- **Frequency ranges may be expressed in terms of decades or octaves.**

  Decade : Frequency band from ω to 10ω.

  Octave : Frequency band from ω to 2ω.

BODE PLOT

Gain Factor K.

- The gain factor multiplies the overall gain by a constant value for all frequencies.
- It has no effect on phase.

$$G(s) = K$$

$$G(j\omega) = K$$

$$M = 20\log|G(j\omega)|$$

$$= 20\log(K) \text{ [dB]}$$

$$\phi = 0$$

M : magnitude, ϕ : phase.

BODE PLOT

Integral Factor 1/jω – pole at the origin.

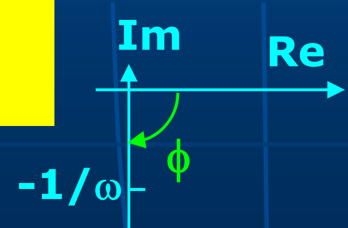
- **Magnitude is a straight line with a slope of -20 dB/decade becoming zero at ω=1 [rad/s].**
- **Phase is constant at -90° at all frequencies.**

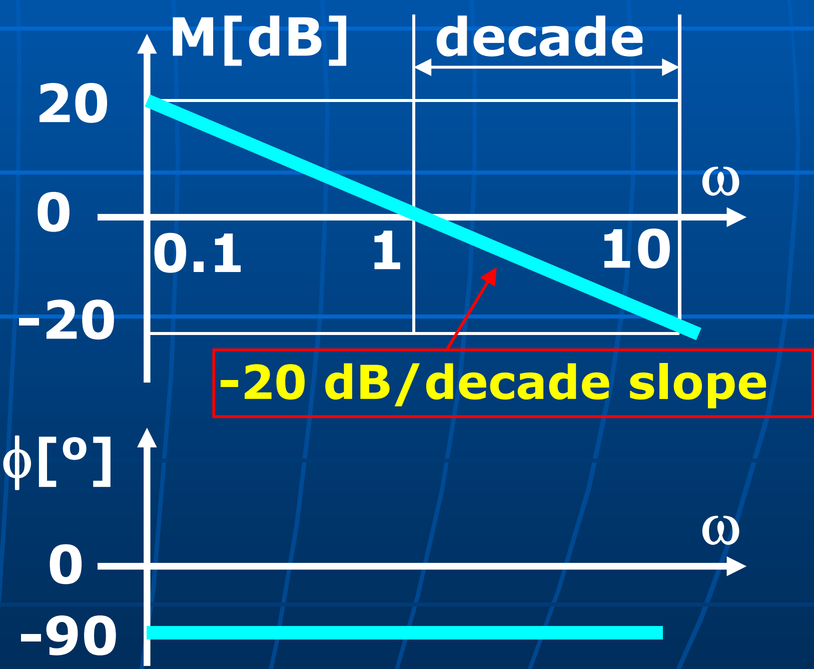
$$G(s)=\frac{1}{s}\ ,\ G(j\omega)=\frac{1}{j\omega}=-\frac{1}{\omega}j$$

$$M=20\log\left|G(j\omega)\right|$$

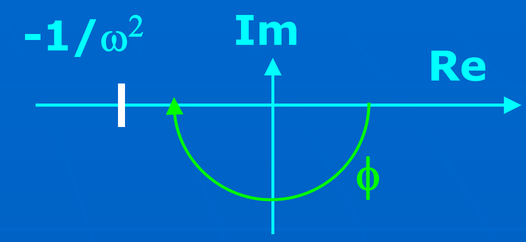
$$=20\log\left(\frac{1}{\omega}\right)=-20\log\omega$$

$$\phi=-90^{o}$$





BODE PLOT



Double pole at the origin.

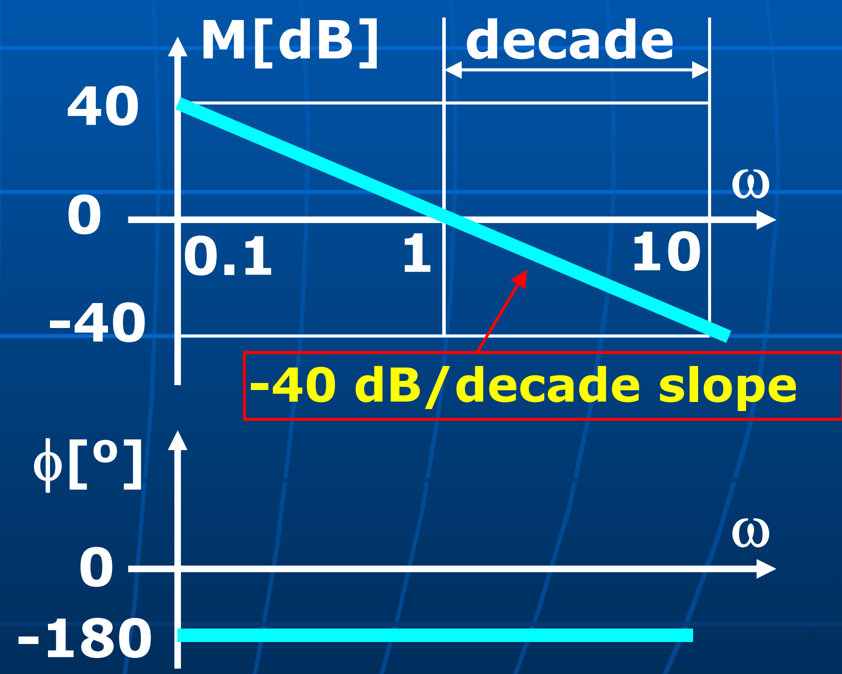
- Simply double the slope of the magnitude and the phase, i.e., -40 dB/decade becoming zero at ω=1 [rad/s] and -180° phase.

$$G(s)=\frac{1}{s^2}\ ,\ G(j\omega)=\frac{1}{(j\omega)^2}=-\frac{1}{\omega^2}$$

$$M=20\log|G(j\omega)|$$

$$=20\log\left(\frac{1}{\omega^2}\right)=-40\log\omega$$

$$\phi=-180^o$$



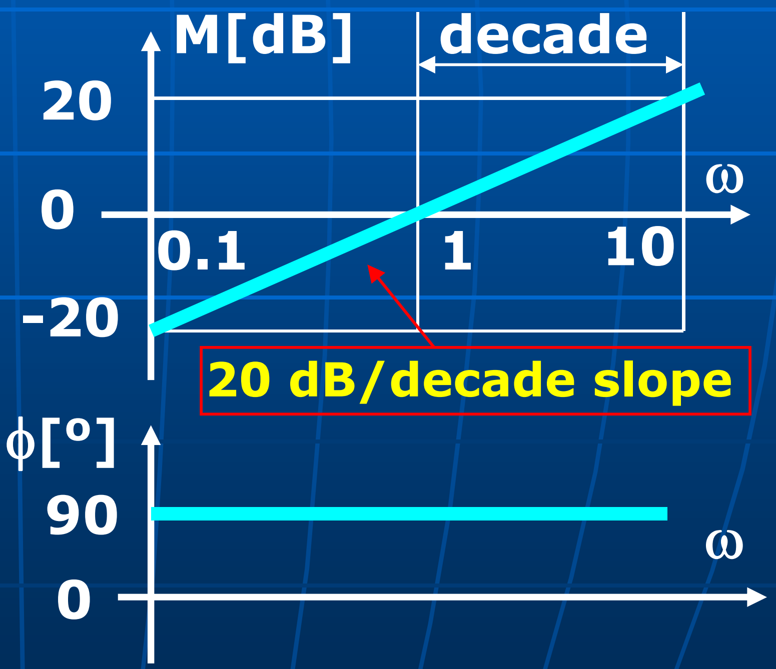
BODE PLOT

Derivative Factor jω – zero at the origin.

- **Magnitude is a straight line with a slope of 20 dB/decade becoming zero at ω=1 [rad/s].**
- **Phase is constant at 90° at all frequencies.**

$$G(s) = s \ , \ G(j\omega) = \omega j$$

$$M = 20\log\left|G(j\omega)\right|$$
$$= 20\log(\omega)$$
$$\phi = 90^{o}$$



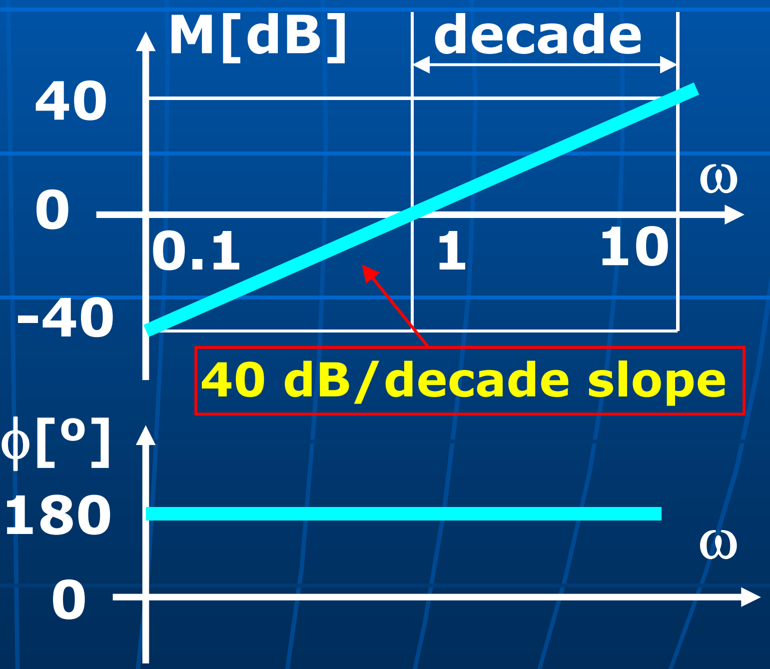
BODE PLOT

Double zero at the origin.

- **Simply double the slope of the magnitude and the phase, i.e., 40 dB/decade becoming zero at ω=1 [rad/s] and 180° phase.**

$$G(s) = s^2 \;,\; G(j\omega) = -\omega^2$$

$$M = 20\log|G(j\omega)|$$
$$= 40\log(\omega)$$
$$\phi = 180^o$$



BODE PLOT – First Order Factor

Simple lag (Real pole) $1/(1+j\omega T)$.

$$G(s)=\frac{1}{1+Ts}$$

$$G(j\omega)=\frac{1}{1+j\omega T}\,\frac{1-j\omega T}{1-j\omega T}=\frac{1}{1+\omega^2T^2}-\frac{\omega T}{1+\omega^2T^2}j$$

$$M=20\log\left|G(j\omega)\right|=20\log\left(\frac{1}{\sqrt{1+\omega^2T^2}}\right)$$

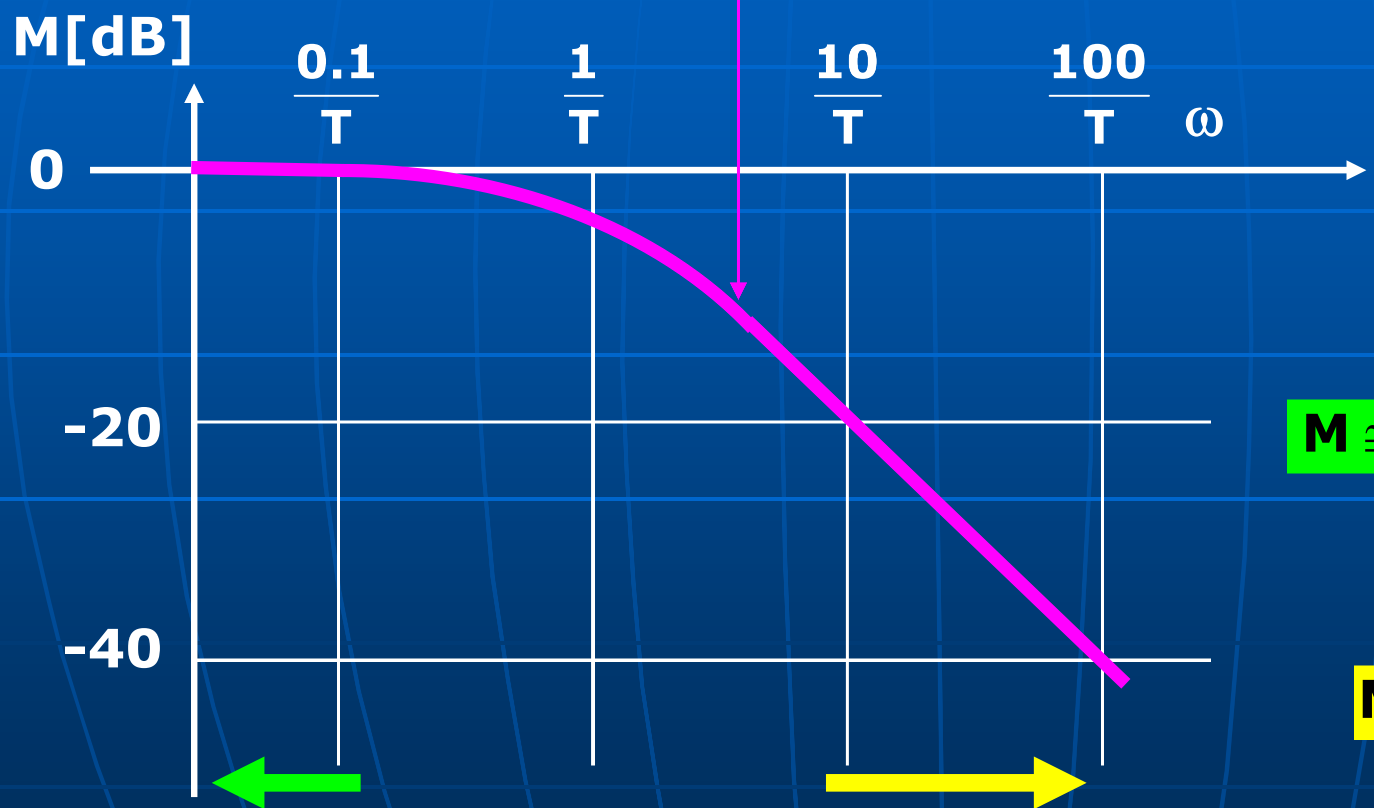
$$M=-20\log\sqrt{1+\omega^2T^2}\ [\text{dB}]$$

$$\phi=\tan^{-1}\left(-\omega T\right)=-\tan^{-1}\omega T$$

BODE PLOT – First Order Factor

Simple lag (Real Pole) $1/(1+j\omega T)$.

$$M = -20\log\sqrt{1+\omega^2 T^2}\ [dB]$$



For $\omega \ll \frac{1}{T}$

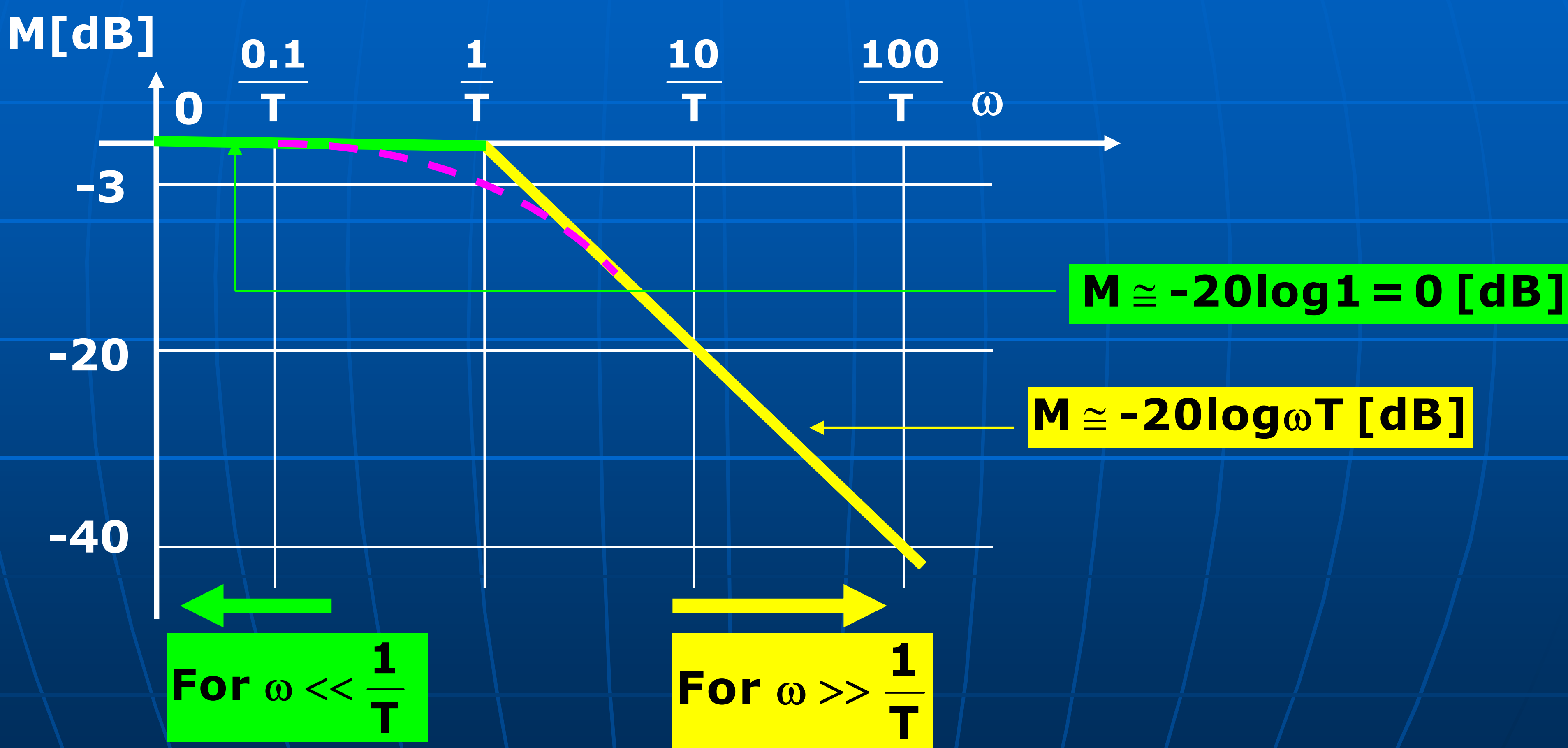
$$M \cong -20\log 1 = 0\ [dB]$$

For $\omega \gg \frac{1}{T}$

$$M \cong -20\log \omega T\ [dB]$$

BODE PLOT – First Order Factor

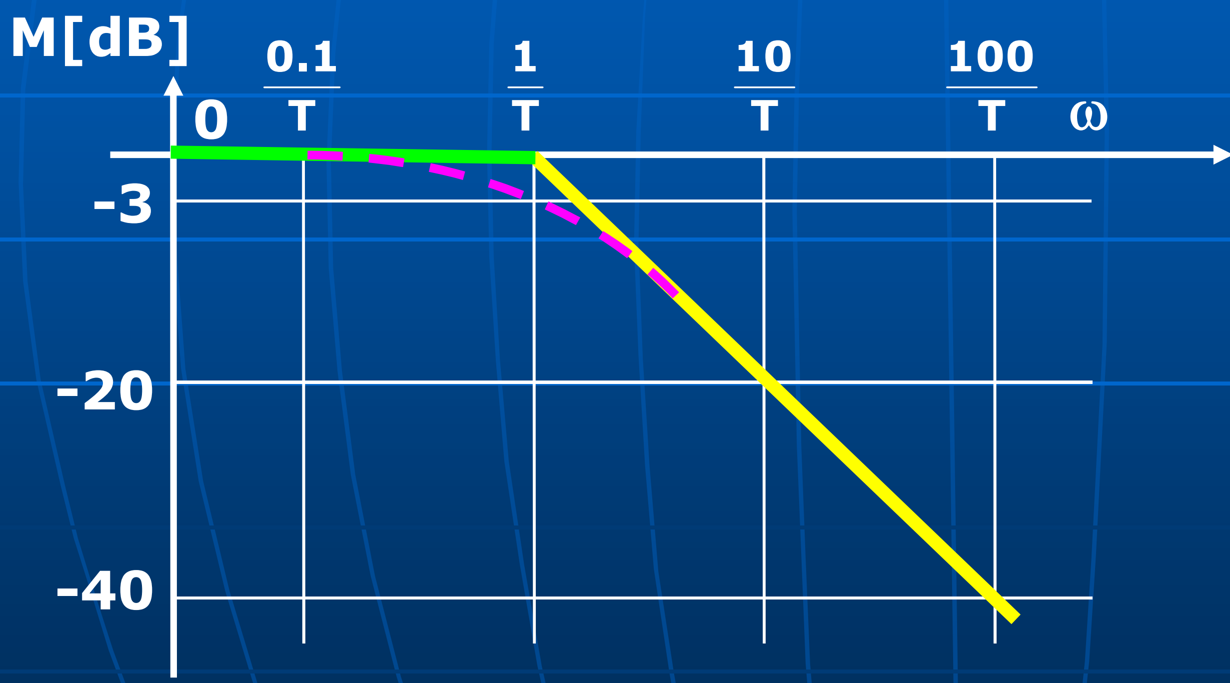
It is clear that the actual magnitude curve can be approximated by two straight lines.



BODE PLOT – First Order Factor

$\omega_c = 1/T$ is called the corner (break) frequency. Maximum error between the linear approximation and the exact value will be at the corner frequency.

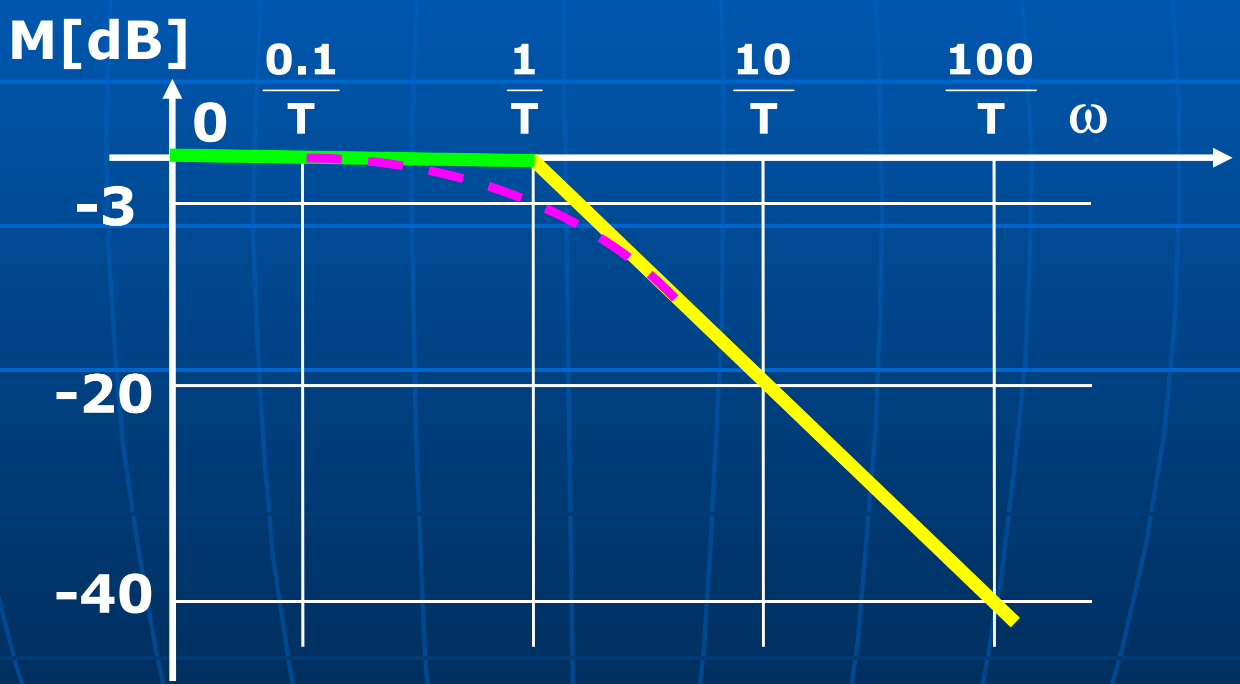
$$M = -20\log\sqrt{1+\omega^2 T^2}\,[dB]$$



$$M\left(\omega = \frac{1}{T}\right) = -20\log\sqrt{2} \cong -3\,[dB]$$

BODE PLOT – First Order Factor

ω	$0.1\omega_c$	$0.5\omega_c$	ω_c	$2\omega_c$	$10\omega_c$
Error [dB]	0.04	1	3	1	0.04



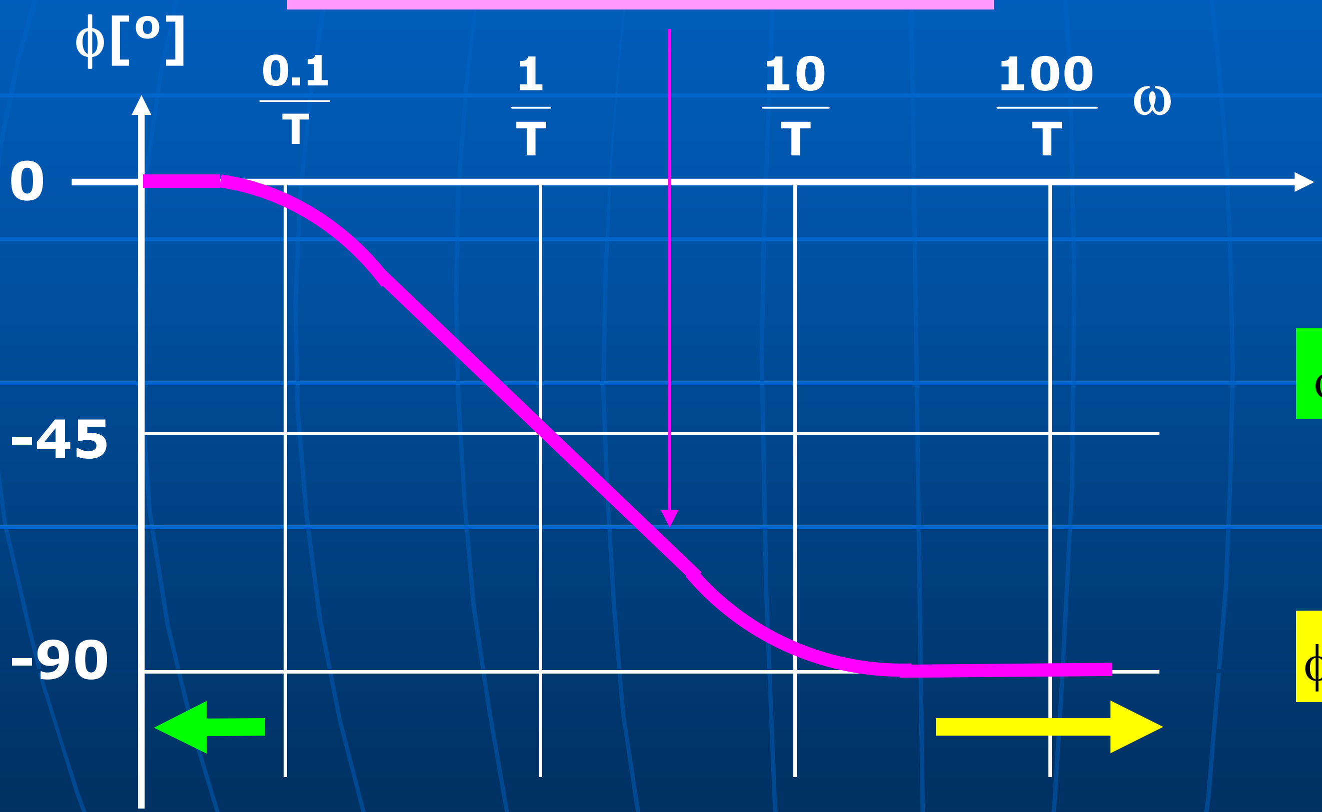
BODE PLOT – First Order Factor

- **Transfer function G(s)=1/(1+Ts) is a low pass filter.**
- **At low frequencies the magnitude ratio is almost one, i.e., the output can follow the input.**
- **For higher frequencies, however, the output cannot follow the input because a certain amount of time is required to build up output magnitude (time constant!).**
- **Thus, the higher the corner frequency the faster the system response will be.**

BODE PLOT – First Order Factor

Simple lag 1/(1+jωT).

$$\phi = \tan^{-1}(-\omega T) = -\tan^{-1}\omega T$$

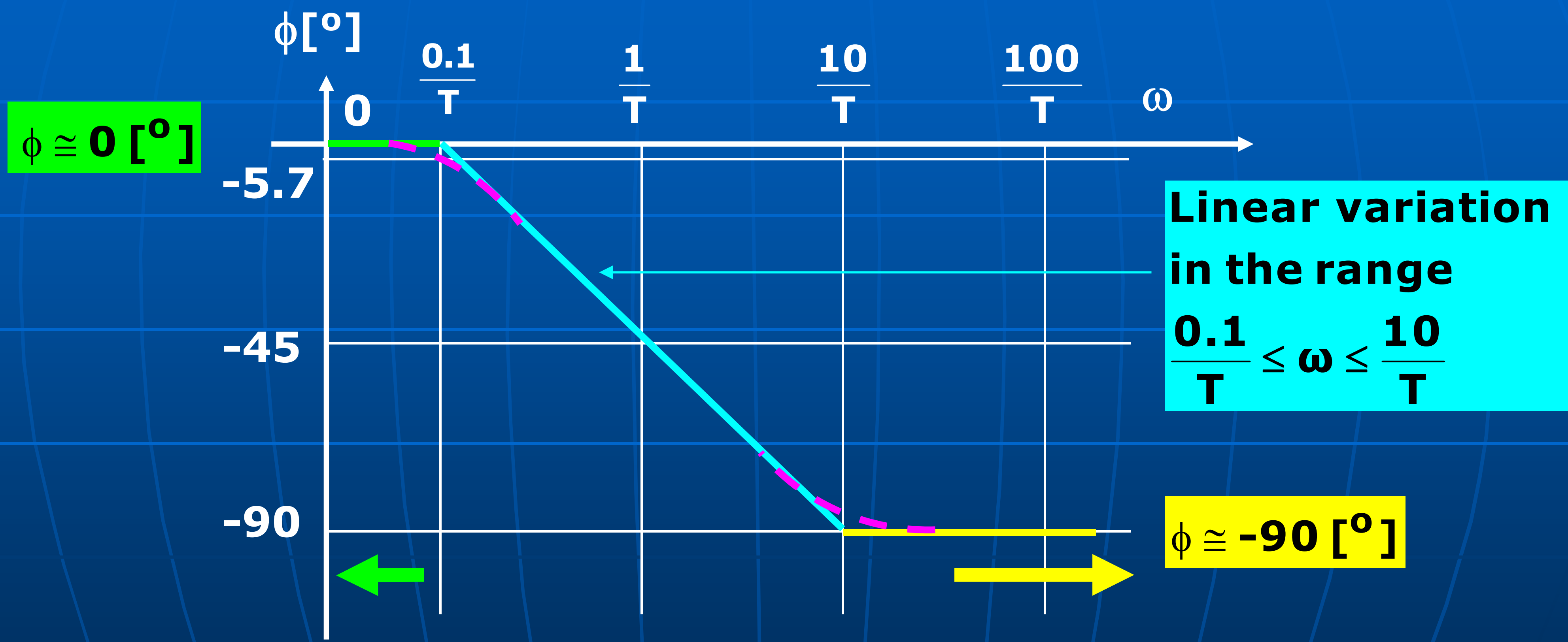


For $\omega << \frac{0.1}{T}$ → $\phi \cong 0\ [^o]$

For $\omega >> \frac{10}{T}$ → $\phi \cong -90\ [^o]$

BODE PLOT – First Order Factor

It is clear that the actual phase curve can be approximated by three straight lines.



In this case corner frequencies are : 0.1/T and 10/T

BODE PLOT – First Order Factor

ω	$0.01\omega_c$	$0.1\omega_c$	ω_c	$10\omega_c$	$100\omega_c$
ϕ [°]	-0.57	-5.7	-45	-84.3	-89.4
Error [°]	0.6	5.7	0	-5.7	-0.6

Thus the maximum error of the linear approximation is 5.7°.

BODE PLOT – First Order Factor

Simple lead (Real zero) 1+jωT.

$$G(s)=1+Ts$$

$$G(j\omega)=1+\omega Tj$$

$$M=20\log\left|G(j\omega)\right|=20\log\left(\sqrt{1+\omega^2T^2}\right)$$

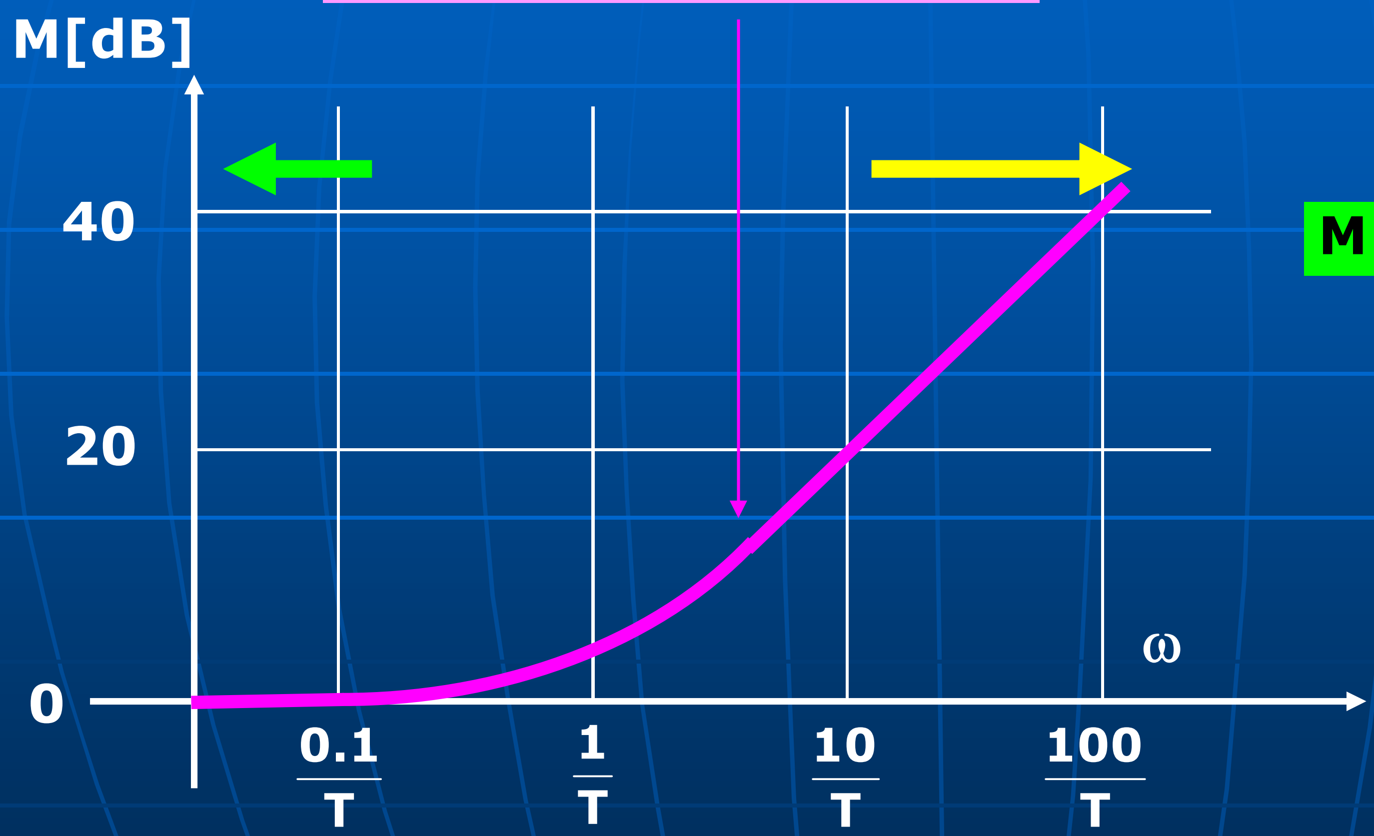
$$M=20\log\sqrt{1+\omega^2T^2}\ [dB]$$

$$\phi=\tan^{-1}(\omega T)=\tan^{-1}\omega T$$

BODE PLOT – First Order Factor

Simple lead (Real zero) $1+j\omega T$.

$$M = 20\log\sqrt{1+\omega^2 T^2}\ [dB]$$



For $\omega << \frac{1}{T}$

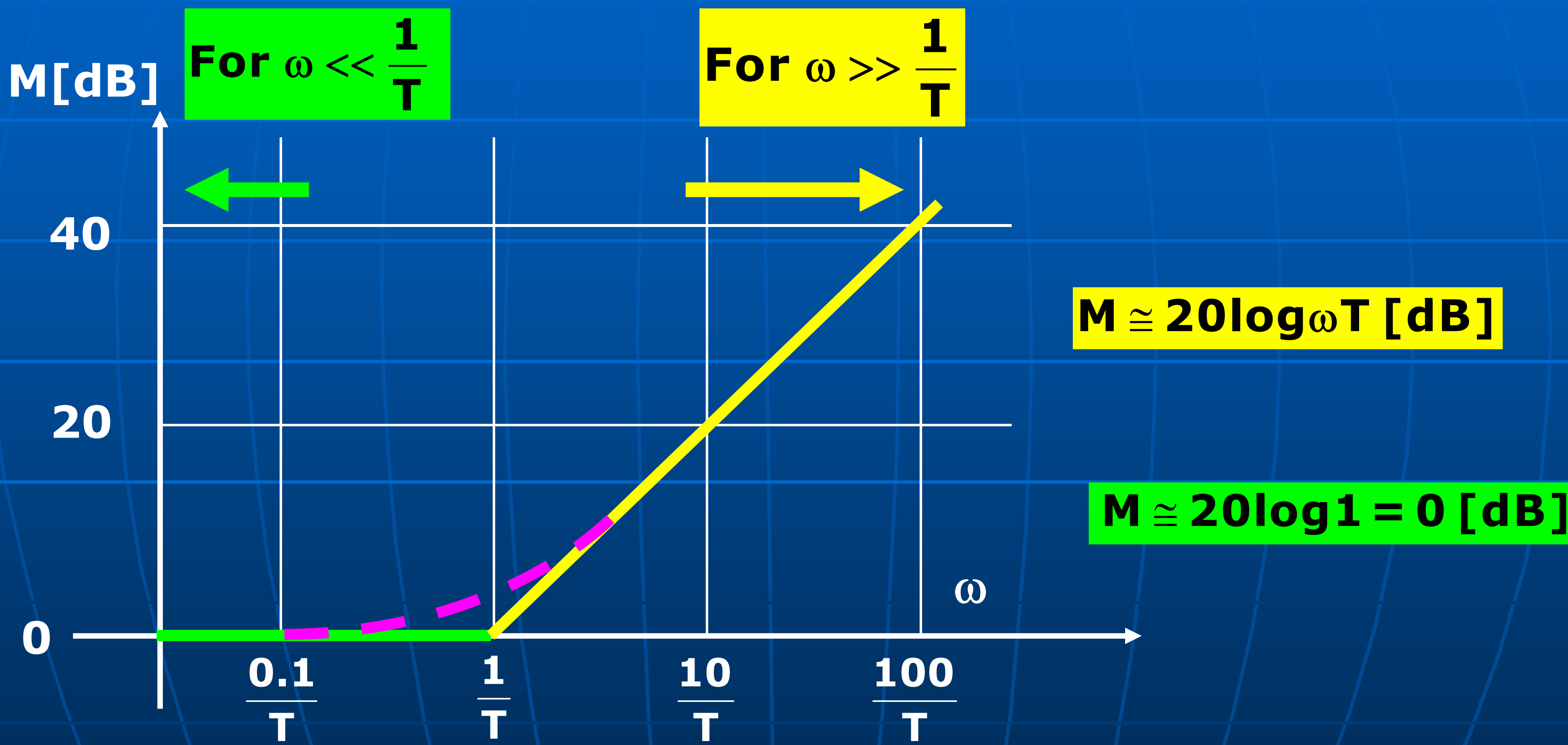
$$M \cong 20\log 1 = 0\ [dB]$$

For $\omega >> \frac{1}{T}$

$$M \cong 20\log \omega T\ [dB]$$

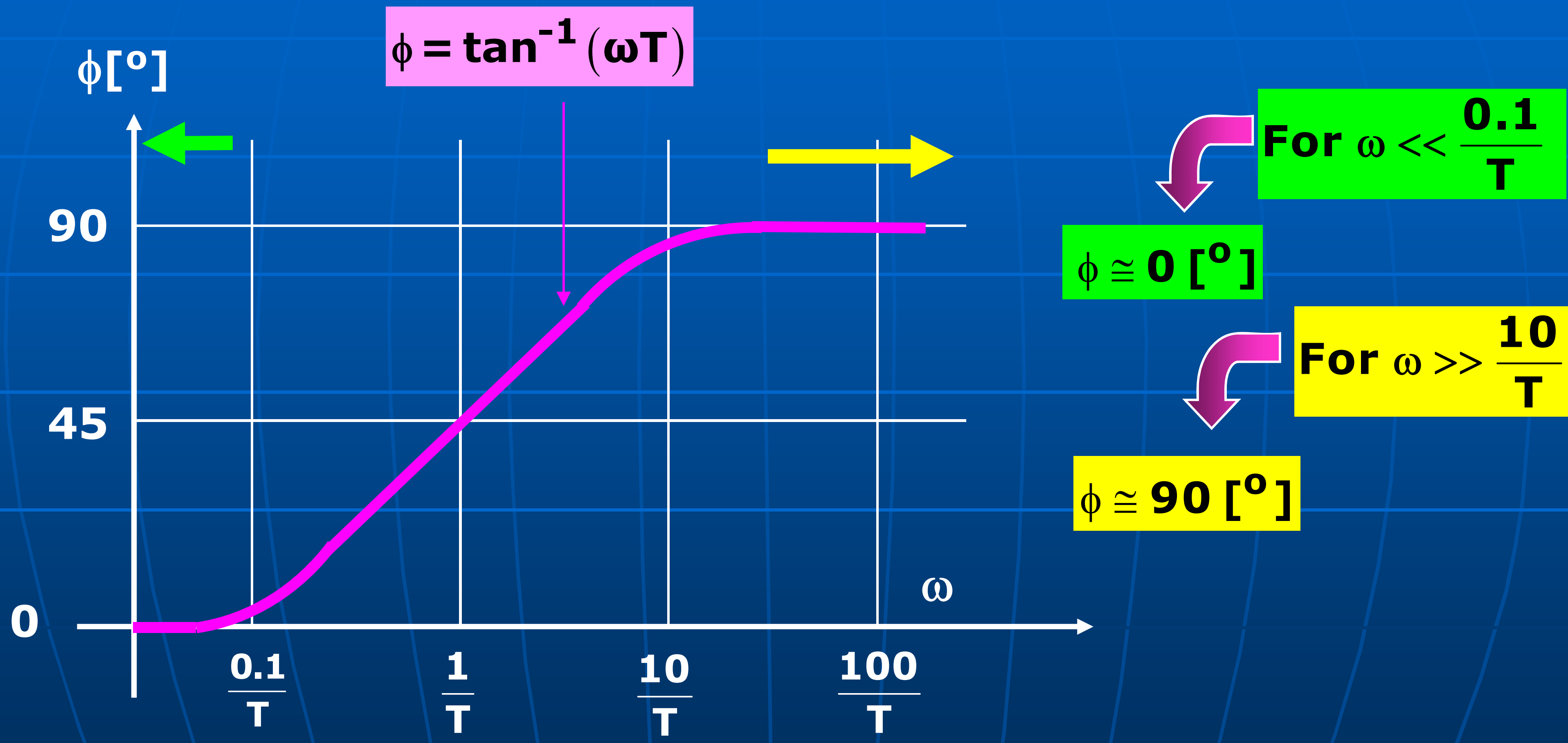
BODE PLOT – First Order Factor

It is clear that the actual magnitude curve can be approximated by two straight lines.



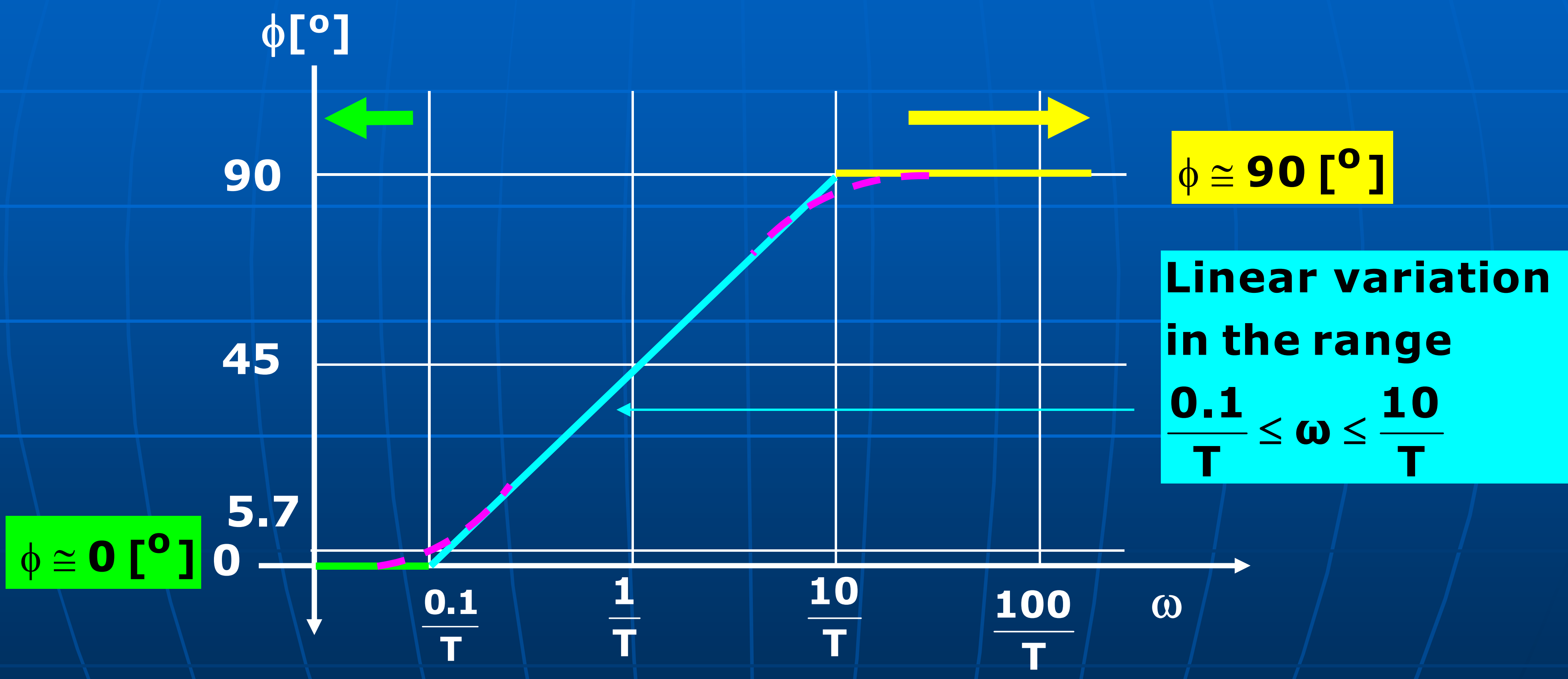
BODE PLOT – First Order Factor

Simple lead $1+j\omega T$.



BODE PLOT – First Order Factor

It is clear that the actual phase curve can be approximated by three straight lines.



BODE PLOT – Quadratic Factors

As overdamped systems can be replaced by two first order factors, only underdamped systems are of interest here.

$$G(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

A set of two complex conjugate poles.

$$G(j\omega)=\frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2+2\xi\left(j\frac{\omega}{\omega_n}\right)+1}$$

$$M=20\log\left|G(j\omega)\right|=-20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}\quad[\text{dB}]$$

BODE PLOT – Quadratic Factors

$$M = 20\log\left|G(j\omega)\right| = -20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2} \quad [dB]$$

Low frequency asymptote, $\omega << \omega_n$:

$$M \cong -20\log(1) = 0 \ [dB]$$

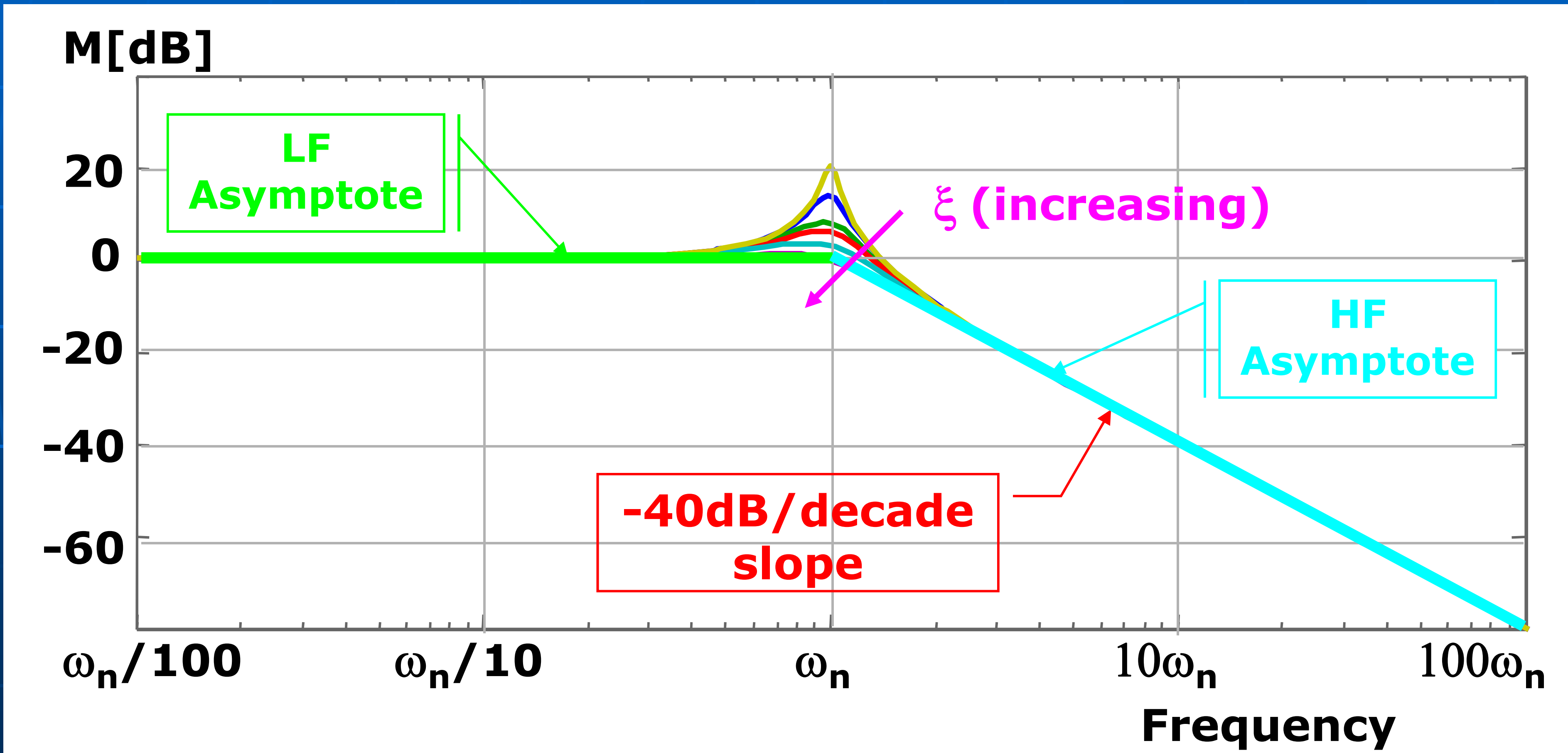
High frequency asymptote, $\omega >> \omega_n$:

$$M \cong -20\log\left(\frac{\omega}{\omega_n}\right)^2 = -40\log\left(\frac{\omega}{\omega_n}\right) \ [dB]$$

Low and high frequency asymptotes intersect at $\omega = \omega_n$, i.e. corner frequency is ω_n.

BODE PLOT – Quadratic Factors

Therefore the actual magnitude curve can be approximated by two straight lines.



BODE PLOT – Quadratic Factors

$$\phi = \angle G(j\omega) = -\tan^{-1} \frac{2\xi \dfrac{\omega}{\omega_n}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}$$

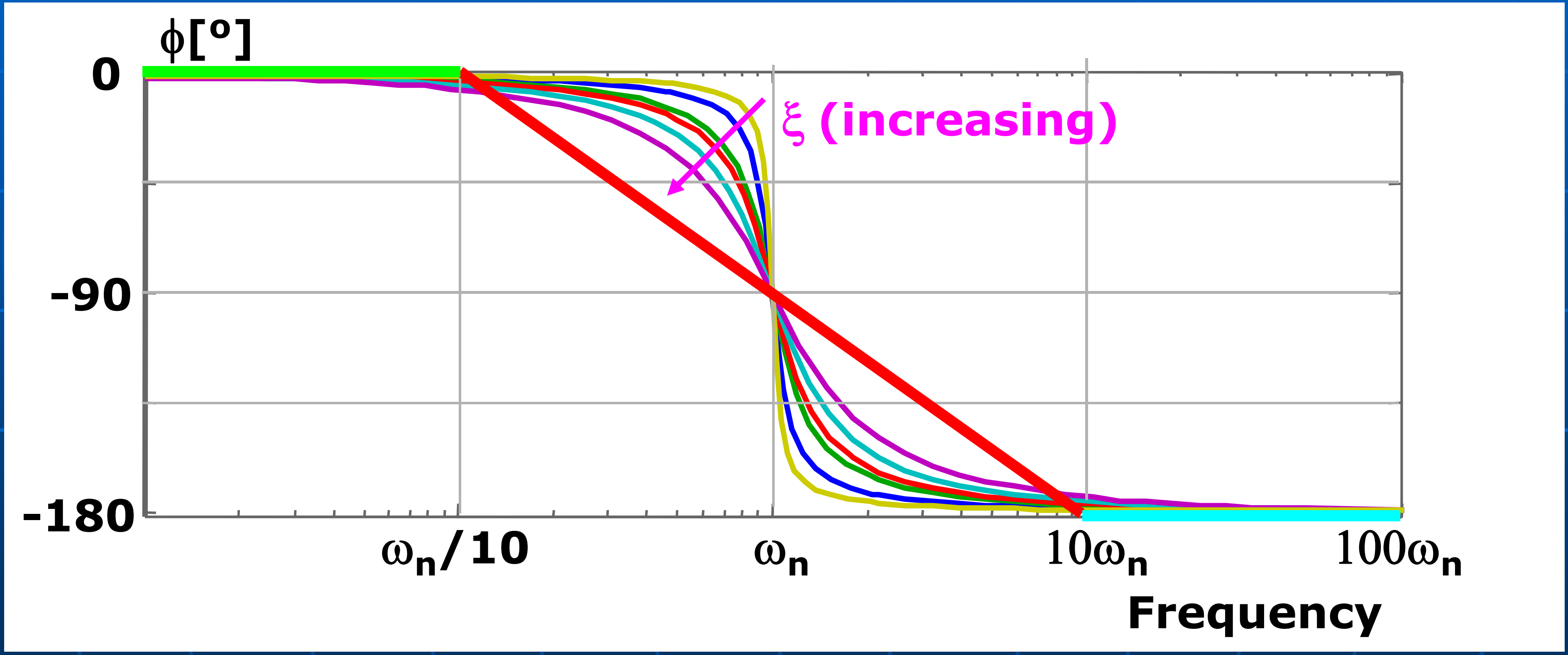
At low frequencies, $\omega \to 0$: $\phi \cong 0\ [^o]$

At $\omega = \omega_n$: $\phi \cong -90\ [^o]$

At high frequencies, $\omega \to$: $\phi \cong -180\ [^o]$

BODE PLOT – Quadratic Factors

Thus, the actual phase curve can be approximated by three straight lines.



Corner frequencies are : $\omega_n/10$ and $10\omega_n$.

BODE PLOT – Quadratic Factors

- **It is observed that, the linear approximations for the magnitude and phase will give more accurate results for damping ratios closer to 1.0.**

- **The peak magnitude is given by :**

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

- **The resonant frequency :**

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

BODE PLOT – Quadratic Factors

- **For ξ=0.707 :**

  M_r=1 (or M=20log1=0 dB).

  Thus, there will be no peak on the magnitude plot.

- **Note the difference that in transient response for step input, there will be no overshoot for critically or overdamped systems, i.e., for $\xi \geq 1.0$.**

BODE PLOT – Example 1a

- **Sketch the Bode plots for the given open loop transfer function of a control system.**

$$T(s)=\frac{100000(1+s)}{s(s+10)\left(0.1s^2+14s+1000\right)}$$

- **First convert to standard form.**

$$T(j\omega)=\frac{100000(1+j\omega)}{(j\omega)(10)(1+0.1\omega j)(1000)\left[\left(j\frac{\omega}{100}\right)^2+1.4\left(j\frac{\omega}{100}\right)+1\right]}$$

$$T(j\omega)=\frac{10(1+j\omega)}{(j\omega)(1+0.1\omega j)\left[\left(j\frac{\omega}{100}\right)^2+1.4\left(j\frac{\omega}{100}\right)+1\right]}$$

BODE PLOT – Example 1b

$$T(j\omega) = \frac{10(1+j\omega)}{(j\omega)(1+0.1\omega j)\left[\left(j\frac{\omega}{100}\right)^2 + 1.4\left(j\frac{\omega}{100}\right) + 1\right]}$$

- Identify the basic factors and corner frequencies :

  -Constant gain K : K=10, 20log10=20 [dB]

  -First order factor (simple lead – real zero) :
  T=1 (ω_{c1}=1/T=1) - for magnitude plot

  -Integral factor : 1/jω

  -First order factor (simple lag – real pole) :
  T=0.1 (ω_{c1}=1/T=10) - for magnitude plot

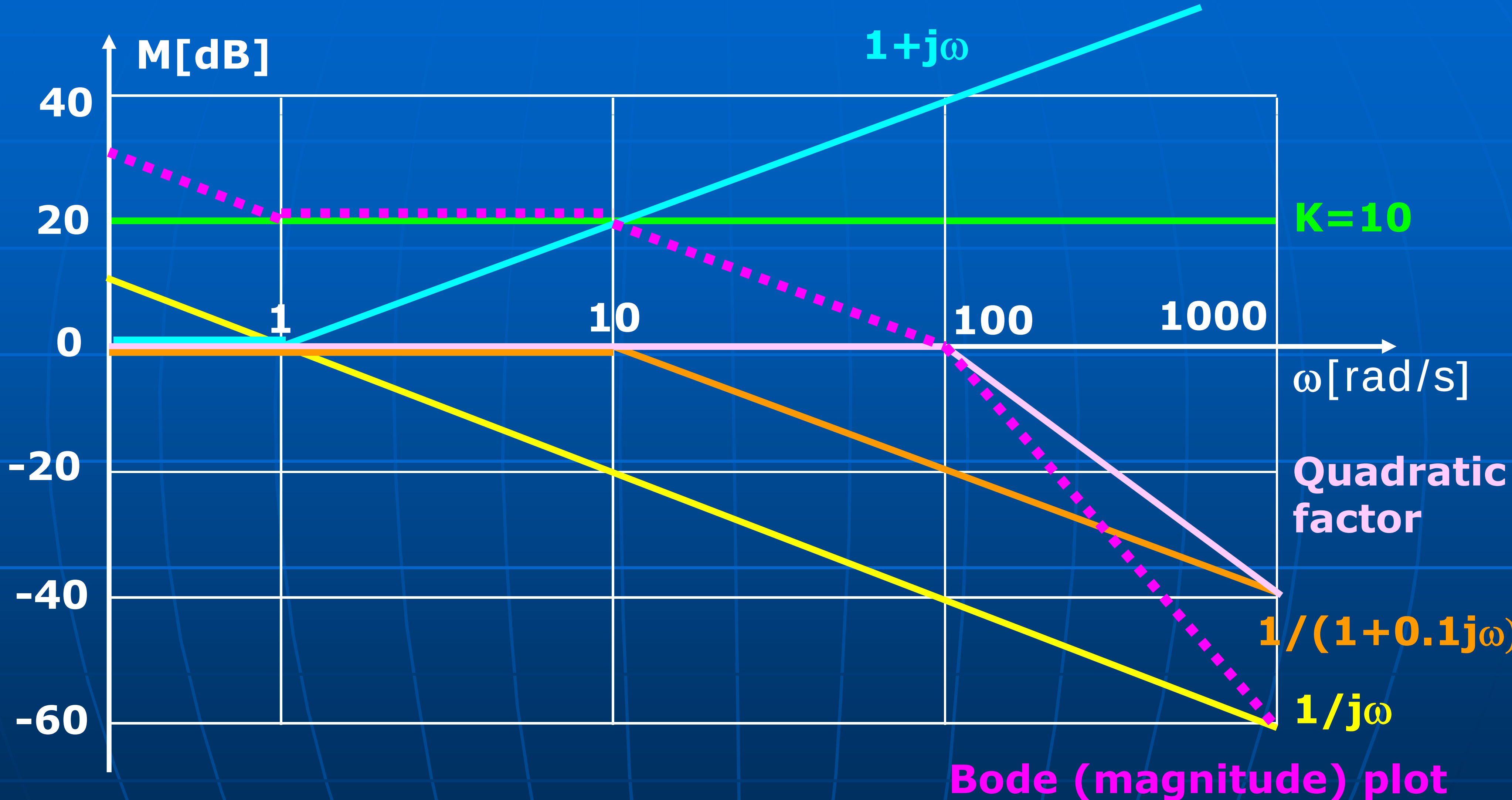
  -Quadratic factor (complex conjugate poles) :
  $\omega_n=\omega_{c1}$=100, ξ=0.7 - for magnitude plot

BODE PLOT – Example 1c

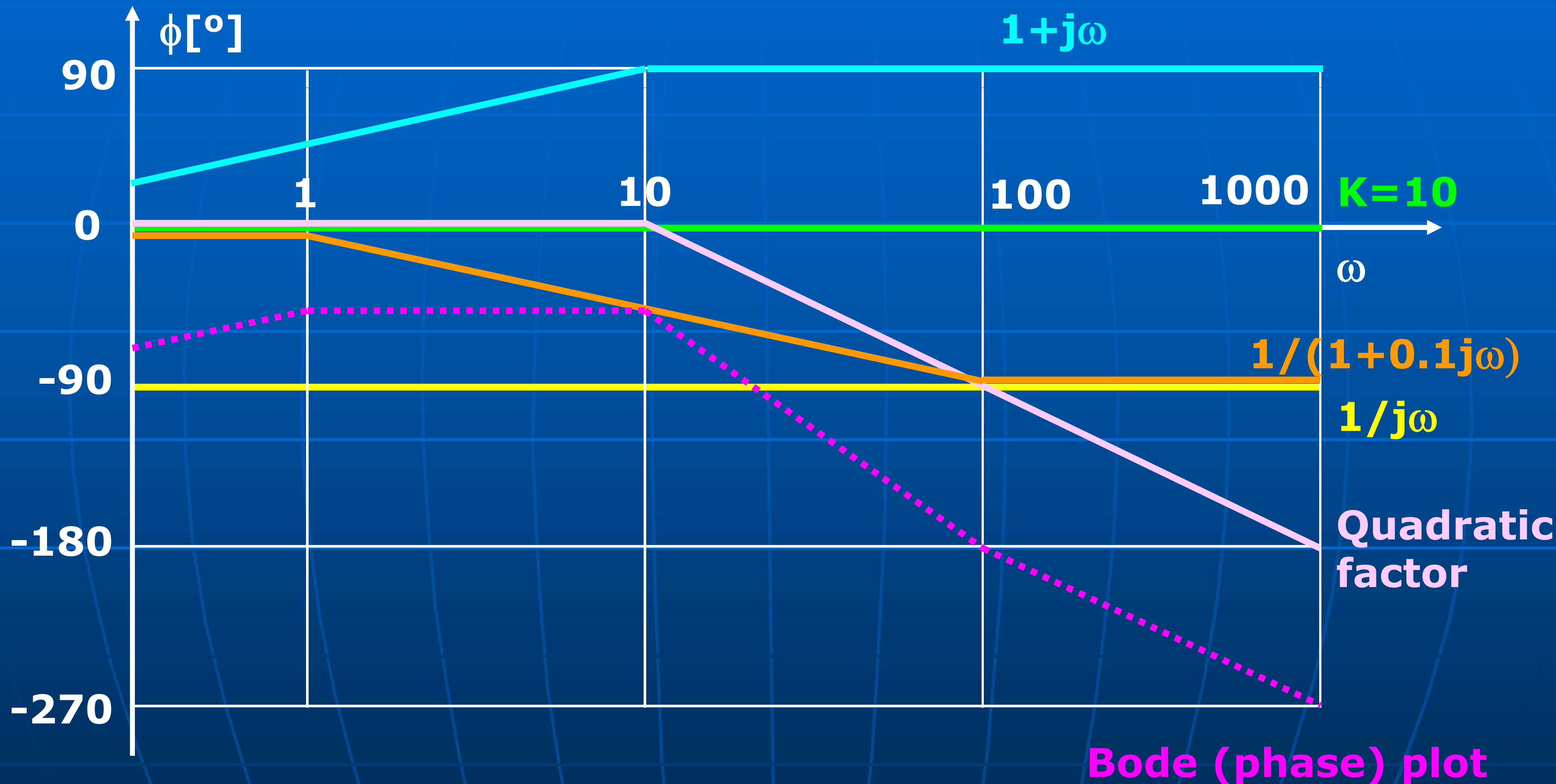
$$T(j\omega)=\frac{10(1+j\omega)}{(j\omega)(1+0.1\omega j)\left[\left(j\frac{\omega}{100}\right)^2+1.4\left(j\frac{\omega}{100}\right)+1\right]}$$

- **Identify the basic factors and corner frequencies :**
  - **-Constant gain K : K=10, 20log10=20 [dB]**
  - **-First order factor (simple lead – real zero) : T=1**
    (ω_{c2}=0.1/T=0.1, ω_{c3}=10/T=10) – for phase plot
  - **-Integral factor : 1/jω**
  - **-First order factor (simple lag – real pole) : T=0.1**
    (ω_{c2}=0.1/T=1, ω_{c3}=10/T=100) – for phase plot
  - **-Quadratic factor (complex conjugate poles) : ω_n=100**
    (ω_{c2}=ω_n/10=10, ω_{c3}=10ω_n=1000) – for phase plot

BODE PLOT – Example 1d



BODE PLOT – Example 1e



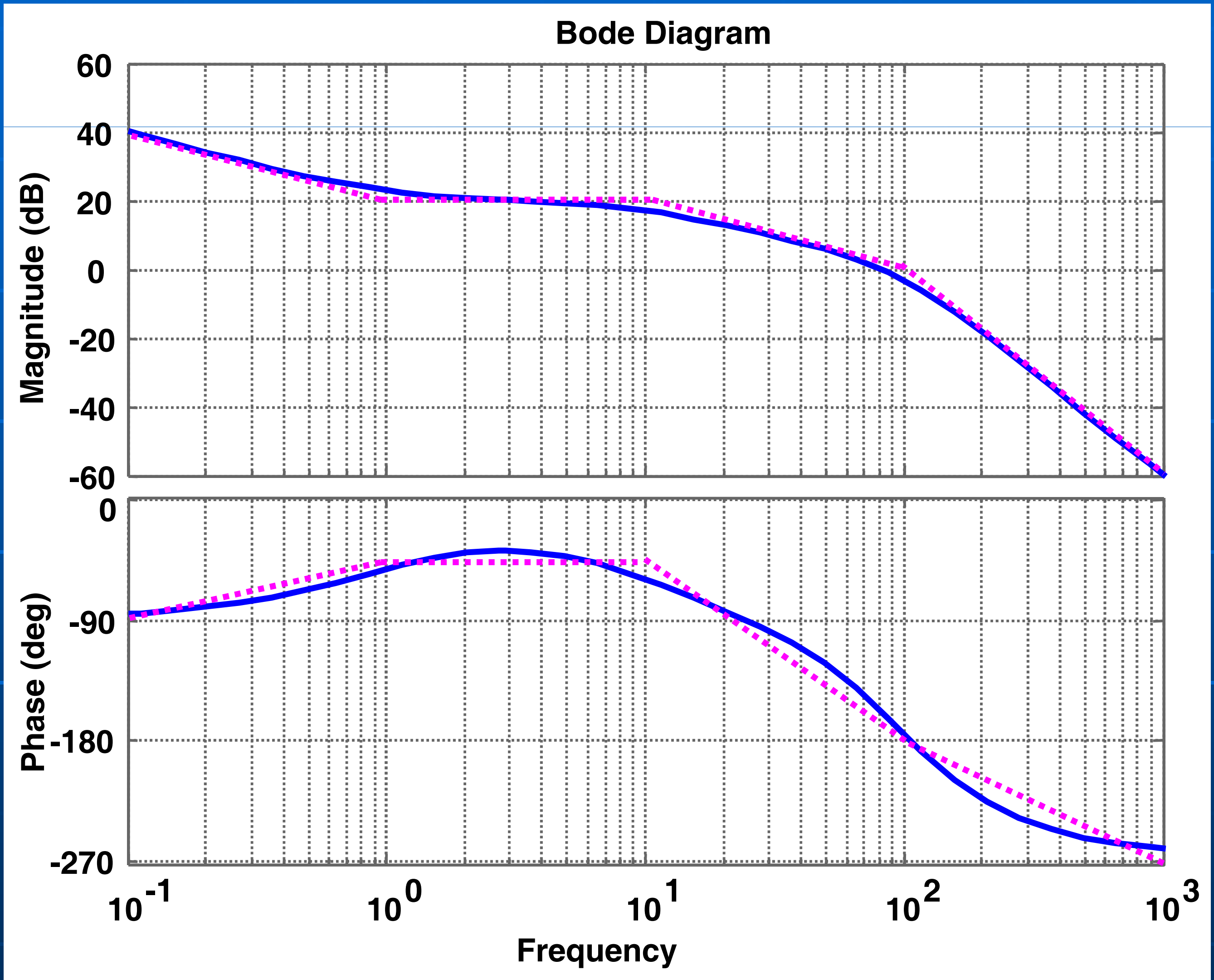
BODE PLOT – Example 1f

- **Matlab plot:** full blue lines

  (just 4 lines to plot !)

```
num=[100000 100000]
den=[0.1 15 1140 10000 0]
bode(num,den)
grid
```

Approximate plots: dashed lines



STABILITY ANALYSIS

Nise Sect. 10.7, pp.638-641

- **Transfer functions which have no poles or zeroes on the right hand side of the complex plane are called minimum phase transfer funtions.**
- **Nonminimum phase transfer functions, on the other hand, have zeros and/or poles on the right hand side of the complex plane.**
- **The major disadvantage of Bode Plot is that stability of <u>only</u> minimum phase systems can be determined using Bode plot.**

STABILITY ANALYSIS

- **From the characteristic equation :**

$$1 + G(s)H(s) = 0 \quad \text{or} \quad G(s)H(s) = -1$$

Then the magnitude and phase for the open loop transfer function become :

$$20\log|G(j\omega)H(j\omega)| = 20\log 1 = 0\,dB$$

$$\angle G(j\omega)H(j\omega) = -180^{o}$$

Thus, when the magnitude and the phase angle of a transfer function are 0 dB and -180°, respectively, then the system is marginally stable.

STABILITY ANALSIS

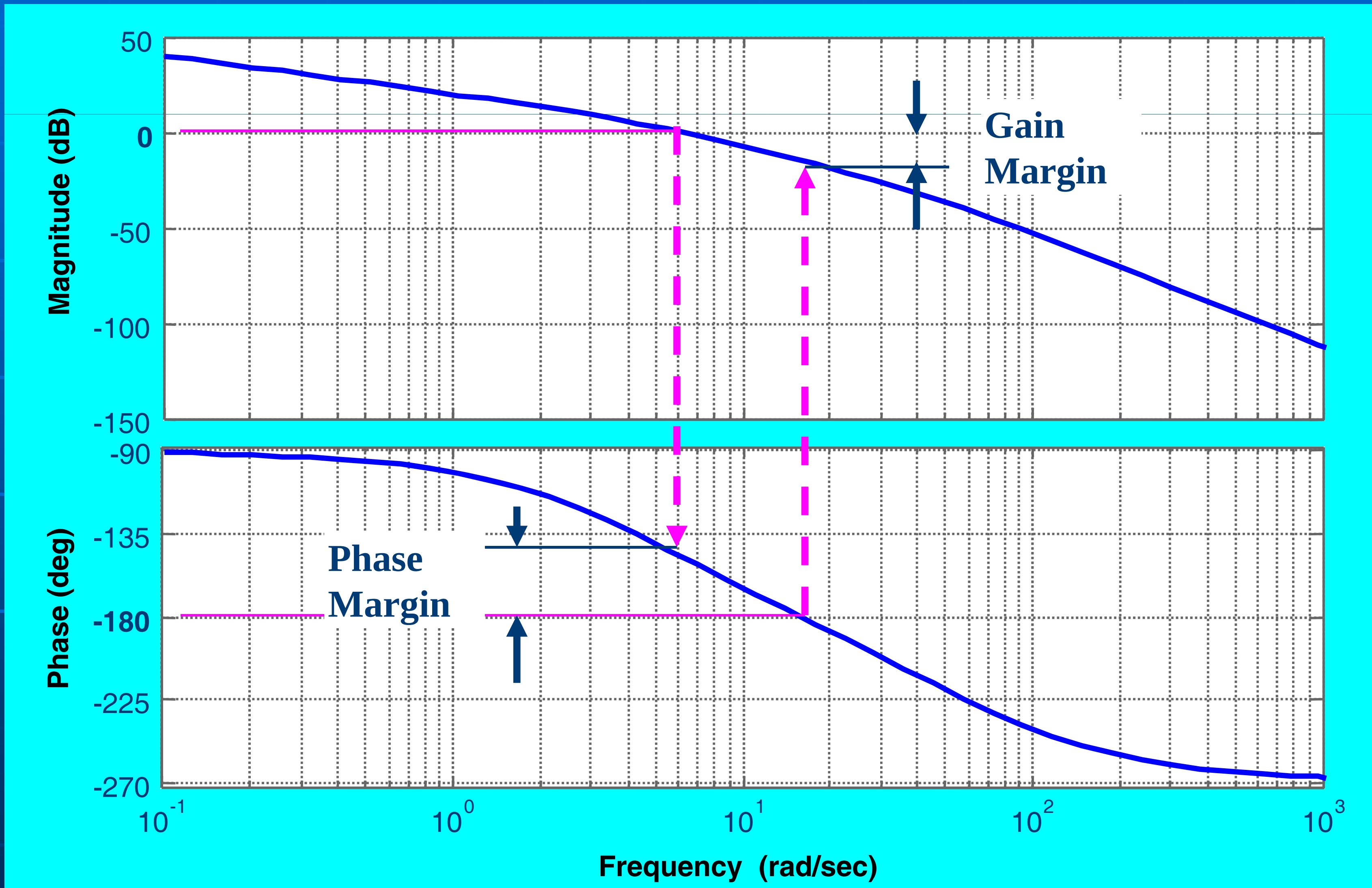
- If at the frequency, for which phase becomes equal to -180º, gain is below 0 dB, then the system is stable (unstable otherwise).
- Further, if at the frequency, for which gain becomes equal to zero, phase is above -180º, then the system is stable (unstable otherwise).
- Thus, <u>relative stability</u> of a minimum phase system can be determined according to these observations.

GAIN and PHASE MARGINS

Nise pp. 638-641

- **Gain Margin :** Additional gain to make the system marginally stable at a frequency for which the phase of the open loop transfer function passes through -180°.
- **Phase Margin :** Additional phase angle to make the system marginally stable at a frequency for which the magnitude of the open loop transfer function is 0 dB.

GAIN and PHASE MARGINS



BODE PLOT

- **Can you identify the transfer function approximately if the measured Bode diagram is available ?**