M E T U
Department of Mathematics

	Elementary Number Theory II Midterm 2	
Code : *Math 366* Acad. Year : *2018-2019* Semester : *Spring* Instructor : *Tolga Karayayla* Date : *24.04.2019* Time : *17.40* Duration : *120 minutes*	Last Name : First Name : Student ID : Department : Signature : 6 Questions on 4 Pages SHOW DETAILED WORK!	

1	2	3	4	5	6		

1. (10+10 pts.) a) Calculate $x = [1;3,\overline{5,2}] = [1;3,5,2,5,2,5,2,...]$.

$x = 1 + \cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{\ddots}}}}} = 1 + \cfrac{1}{3+\cfrac{1}{y}}$ where $y = 5 + \cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{2+}}} = 5 + \cfrac{1}{2+\cfrac{1}{y}}$

so,

$y = 5 + \frac{y}{2y+1} \Rightarrow y = \frac{11y+5}{2y+1}$

$2y^2 - 10y - 5 = 0$

$y = \frac{10 \pm \sqrt{140}}{4}$

$[y] = 5 \Rightarrow y \geq 5$

$\Rightarrow y = \frac{5+\sqrt{35}}{2}$

$y = \frac{5+\sqrt{35}}{2} \Rightarrow x = 1 + \cfrac{1}{3+\cfrac{1}{y}}$

$= 1 + \frac{y}{3y+1} = \frac{4y+1}{3y+1} = \frac{4\frac{5+\sqrt{35}}{2}+1}{3\cdot\frac{5+\sqrt{35}}{2}+1} = \frac{22+4\sqrt{35}}{17+3\sqrt{35}}$

b)Find the infinite continued fraction representation of $x = \frac{13+\sqrt{5}}{4}$.

$x_0 = x$, $a_0 = [x_0] = \left[\frac{13+\sqrt{5}}{4}\right] = 3$

$x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\frac{13+\sqrt{5}}{4} - 3} = \frac{4}{\sqrt{5}+1} = \sqrt{5} - 1$, $a_1 = [x_1] = [\sqrt{5} - 1] = 1$

$x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\sqrt{5}-2} = \sqrt{5} + 2$, $a_2 = [x_2] = [\sqrt{5}+2] = 4$

$x_3 = \frac{1}{x_2 - a_2} = \frac{1}{\sqrt{5}+2-4} = \frac{1}{\sqrt{5}-2} = \sqrt{5} + 2$, $a_3 = [x_3] = 4$

We obtained $x_2 = x_3$, then we got $x_2 = x_3 = x_4 = x_5 = \dots = x_n$ for $n \geq 3$

$4 = a_2 = a_3 = \dots = a_n$ for $n \geq 3$

$x_{k+1} = \frac{1}{x_k - [x_k]}$ for $k \geq 0$

$x_k = x_{k+1} \Rightarrow x_{k+2} = x_{k+1}$

So by induction, $x_2 = x_3 \Rightarrow x_2 = x_n$ for $n \geq 3$.

Therefore, $\frac{13+\sqrt{5}}{4} = [a_0; a_1, a_2, a_3, \dots] = [3; 1, 4, 4, 4 \dots] = [3; 1, \overline{4}]$

2. (10+10 pts.) a) Using the information $\sqrt{22} = [4; \overline{1, 2, 4, 2, 1, 8}]$ find the fundamental solution of the equation $x^2 - 22y^2 = 1$.

$n = 6$: length of repeating block.

All solutions in positive integers are $(x, y) = (p_{nk-1}, q_{nk-1}) = (p_{6k-1}, q_{6k-1})$ for $k \in \mathbb{Z}^+$

where $C_m = \frac{p_m}{q_m}$: m^{th} convergent of $\sqrt{22}$.

$k = 1 \Rightarrow$ Fundamental solution is $(x, y) = (p_5, q_5)$

n	-2	-1	0	1	2	3	4	(5)	6
a_n			4	1	2	4	2	1	8
p_n	0	1	4	5	14	61	136	197	
q_n	1	0	1	1	3	13	29	42	

To fill in the table, we used

$p_{n+1} = a_{n+1} p_n + p_{n-1}$

$q_{n+1} = a_{n+1} q_n + q_{n-1}$

So, fundamental solution is:

$(x, y) = (197, 42)$

b) Find 3 solutions $(x, y) \in \mathbb{Z}_+^2$ of the equation $x^2 - 27y^2 = 1$ (Hint: $\sqrt{27} = [5; \overline{5, 10}]$).

Similar to part a above: Fundamental solution is $(x, y) = (p_{2\cdot 1-1}, q_{2\cdot 1-1})$

$= (p_1, q_1)$

$= (26, 5)$

n	-2	-1	0	(1)	2
a_n			5	5	10
p_n	0	1	5	26	
q_n	1	0	1	5	

Then $x_n + y_n\sqrt{27} = (26 + 5\sqrt{27})^n \Rightarrow (x_n, y_n)$ is a solution.

$n = 2 \Rightarrow (26 + 5\sqrt{27})^2 = 26^2 + 25\cdot 27 + (260)\sqrt{27} \Rightarrow \boxed{(x_2, y_2) = (1351, 260)}$

$= 676 + 675 + 260\sqrt{27}$

$x_3 + y_3\sqrt{27} = (26 + 5\sqrt{27})^3 = (26 + 5\sqrt{27})^2 (26 + 5\sqrt{27})$

$= (1351 + 260\sqrt{27})(26 + 5\sqrt{27})$

$= 1351\cdot 26 + 260\cdot 5\cdot 27 + (26\cdot 260 + 1351\cdot 5)\sqrt{27}$

$\Rightarrow \boxed{(x_3, y_3) = (1351\cdot 26 + 1300\cdot 27,\ 5760 + 6755)}$

3. (15 pts.) Let d be a positive integer which is not a square, and let $k \in \mathbb{Z}$. Show that if $x^2 - dy^2 = k$ has a solution $(x, y) \in \mathbb{Z}^2$, then there are infinitely many solutions $(x, y) \in \mathbb{Z}^2$. (Hint: Use the properties of the norm function $N(x+y\sqrt{d}) = (x+y\sqrt{d})(x-y\sqrt{d})$ on $\mathbb{Z}[\sqrt{d}]$. Consider the numbers which have norm 1).

$N(x+y\sqrt{d}) = (x+y\sqrt{d})(x-y\sqrt{d}) = x^2 - dy^2$

$N(x+y\sqrt{d}) = 1 \iff x^2 - dy^2 = 1$ (Pell's Eq.), it has infinitely many solutions for $d \in \mathbb{Z}^+$ which are not squares.

Let (x_n, y_n) $n \in \mathbb{Z}^+$ be infinitely many distinct solutions of $x^2 - dy^2 = 1$.

Assume $(x,y) = (r,s)$ is a solution of $x^2 - dy^2 = k$, so $r^2 - ds^2 = k$, $N(r+s\sqrt{d}) = k$

Using $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$, we got:

$N((x_n + y_n\sqrt{d}) \cdot (r+s\sqrt{d})) = N(x_n + y_n\sqrt{d}) \cdot N(r+s\sqrt{d}) = 1 \cdot k = k$

$\Rightarrow N(A_n + B_n\sqrt{d}) = k$, $A_n^2 - dB_n^2 = k$ $\Rightarrow$ All $(A_n, B_n) = (x,y)$ are solutions

where

$A_n + B_n\sqrt{d} = (x_n + y_n\sqrt{d})(r+s\sqrt{d})$

$A_n = x_n \cdot r + y_n \cdot s \cdot d$

$B_n = x_n s + y_n r$

Since there are infinitely many (x_n, y_n), there are infinitely many (A_n, B_n).

4. (15 pts.) Find $gcd(10 + 16i, 5 + i)$ in $\mathbb{Z}[i]$ using the Euclidean algorithm.

$\frac{10+16i}{5+i} = \frac{(10+16i)(5-i)}{(5+i)(5-i)} = \frac{66+70i}{26}$

$2 < \frac{66}{26} < 3$, $\left|\frac{66}{26} - 3\right| < \frac{1}{2}$

$2 < \frac{70}{26} < 3$, $\left|\frac{70}{26} - 3\right| < \frac{1}{2}$

$\Rightarrow 10+16i = (5+i)(3+3i) + r_1$

$\Rightarrow r_1 = 10+16i - (5+i)(3+3i)$

$r_1 = 10+16i - (12+18i)$

$r_1 = -2-2i$

$\frac{5+i}{r_1} = \frac{(5+i)(-2+2i)}{(-2-2i)(-2+2i)} = \frac{-12+8i}{8}$

$-2 < -\frac{12}{8} < -1$, $\left|-\frac{12}{8} - (-1)\right| \leq \frac{1}{2}$

$1 \leq \frac{8}{8} < 2$,

$5+i = (-2-2i)(-1+i) + r_2$

$r_2 = 5+i - (-2-2i)(-1+i)$

$r_2 = 5+i - (4+0i)$

$r_2 = 1+i$

$\frac{r_1}{r_2} = \frac{-2-2i}{1+i} = -2 \in \mathbb{Z}[i]$

$\Rightarrow r_2 \mid r_1$ in $\mathbb{Z}[i]$.

So, we got the Division Algorithms:

$10+16i = (5+i)(3+3i) + (-2-2i)$

$5+i = (-2-2i)(-1+i) + (1+i)$

$-2-2i = (1+i)(-2) + 0$

Then by Euclidean Algorithm:

$gcd(10+16i, 5+i) = (1+i)$

(The last nonzero remainder)

5. (15 pts.) Find a prime factorization of $21 - 27i$ in Gaussian integers $\mathbb{Z}[i]$.

$21-27i = 3(7-9i)$ $\quad 3 \equiv 3 \pmod 4$ is an ordinary prime $\Rightarrow 3$ is a Gaussian prime

$N(7-9i) = 7^2+9^2 = 130 = 13 \cdot 2 \cdot 5$

$2 \mid N(7-9i) \Rightarrow 1+i \mid 7-9i$

($N(1+i)=2$, any Gaussian prime π where $N(\pi)=2$ is an associate of $1+i$)

$\frac{7-9i}{1+i} = \frac{(7-9i)(1-i)}{2} = \frac{-2-16i}{2} = -1-8i$, $N(-1-8i) = 65 = 13 \cdot 5$

$\pi = 3+2i \Rightarrow N(\pi) = N(\bar{\pi}) = 13$, $\pi, \bar{\pi}$ are Gaussian primes, $\pi \mid -1-8i \vee \bar{\pi} \mid -1-8i$

$\frac{-1-8i}{3-2i} = \frac{(-1-8i)(3+2i)}{13} = \frac{13-26i}{13} = 1-2i$, $N(1-2i)=5$ (ordinary prime)

so $1+2i$ is a Gaussian prime.

We obtained:

$21-27i = 3 \cdot (1+i)(3-2i)(1-2i)$

where all factors are Gaussian primes.

6. (15 pts.) Show that there are infinitely many odd integers n such that n and $\frac{n-1}{366}$ are both perfect squares.

Let $n = x^2$, $\frac{n-1}{366} = y^2$ for some $x, y \in \mathbb{Z}$ (Are there such $x, y \in \mathbb{Z}$?)

Then $x^2 - 366y^2 = n-(n-1) = 1$, so $(x,y) \in \mathbb{Z}^2$ must be a solution of

$x^2 - 366y^2 = 1$ ($366 > 0$, not a square, so this Pell's eq. has infinitely many solutions in $\mathbb{Z}^2$)

For any $(x,y) \in \mathbb{Z}^2$ satisfying $x^2 - 366y^2 = 1$, $n = x^2 = 1 + 366y^2$ is odd

and $n-1 = 366y^2$ so $\frac{n-1}{366} = y^2$ is a square

Infinitely many such (x,y) gives infinitely many n satisfying

n is odd, n and $\frac{n-1}{366}$ are squares.