Math 366 - Quiz 4

Name and Student ID:

Question (4–3 pts.): a) Show that $I_{10} = \mathbb{Z}[\sqrt{10}]$ is not a UFD. (Hint: Factorize 6 in I_{10}. At some point in your justification, it may help to reduce an equation modulo 10.)
b) Verify that $(3) = (3, 1+\sqrt{10})(3, 1-\sqrt{10})$ in I_{10}.

a) $6 = 2 \cdot 3 = (4+\sqrt{10}) \cdot (4-\sqrt{10})$ in I_{10}.

$N(2) = 4$, $N(3) = 9$, $N(4+\sqrt{10}) = N(4-\sqrt{10}) = (4+\sqrt{10}) \cdot (4+\sqrt{10})'$
$= (4+\sqrt{10})(4-\sqrt{10})$
$= 16 - 10 = \underline{\underline{6}}$

Claim: $2, 3, 4 \pm \sqrt{10}$ are irreducible in I_{10}.

Proof: Let $x \mid 2$ and assume x is not a unit $(|N(x)| \neq 1)$ and x is not an associate of 2, then $2 = xy$

$4 = N(2) = N(x) \cdot N(y)$ $\quad (|N(x)| \neq 1, |N(y)| \neq 1)$

so $|N(x)| = 2$.

Similarly, if we assume $3, 4 \pm \sqrt{10}$ are not irreducible and x is a nonunit, and non associate of 3 or $4 \pm \sqrt{10}$, then

$x \mid 3 \Rightarrow N(x) \mid 9 \Rightarrow |N(x)| = 3$

$x \mid 4 \pm \sqrt{10} \Rightarrow N(x) \mid N(4 \pm \sqrt{10}) = 6 \Rightarrow |N(x)| = 2$ or $N(x) = 3$.

So, if we show that there is no $x \in I_{10}$ such that $N(x) = \pm 2$ or $N(x) = \pm 3$, this will prove that $2, 3$ and $4 \pm \sqrt{10}$ are all irreducible in I_{10}.

Let $x = a + b\sqrt{10}$, then

$N(x) = \pm 2 \Leftrightarrow a^2 - 10b^2 = \pm 2$, so $a^2 \equiv \pm 2 \pmod{10} \Rightarrow$ no solution.

$N(x) = \pm 3 \Leftrightarrow a^2 - 10b^2 = \pm 3$, so $a^2 \equiv \pm 3 \pmod{10} \Rightarrow$ no solution

---

$6 = 2 \cdot 3 = (4+\sqrt{10}) \cdot (4-\sqrt{10})$ product ~~~~ of irreducibles.

2 is not an associate of $4 \pm \sqrt{10}$ (their norms are different)

3 is not an associate of $4 \pm \sqrt{10}$ (their norms are different)

Therefore factorization into a product of irreducibles in I_{10} is not unique up to order of factors and up to considering associate irreducibles the same.

Thus, I_{10} is not a UFD.

6) Let $p_1 = (3, 1+\sqrt{10})$, $p_2 = (3, 1-\sqrt{10})$

$p_1 = \{ 3(a+b\sqrt{10}) + (1+\sqrt{10})(c+d\sqrt{10}) \mid a,b,c,d \in \mathbb{Z} \}$

so $\{3, 3\sqrt{10}, 1+\sqrt{10}, (1+\sqrt{10})\sqrt{10} = 10+\sqrt{10}\}$ span p_1 over $\mathbb{Z}$.

$3\sqrt{10} = 3(1+\sqrt{10}) - 1\cdot 3$ is in the span of $\{3, 1+\sqrt{10}\}$ over $\mathbb{Z}$

$10+\sqrt{10} = 3\cdot 3 + 1\cdot(1+\sqrt{10})$ is in the span of $\{3, 1+\sqrt{10}\}$ over $\mathbb{Z}$.

Hence $\{3, 1+\sqrt{10}\}$ is a $\mathbb{Z}$-basis of the ideal p_1.

Then since $3 = 3\cdot 1 + 0\cdot\sqrt{10}$
$1+\sqrt{10} = 1\cdot 1 + 1\cdot\sqrt{10}$, $N(p_1) = \left| \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} \right| = |3| = 3$ – an ordinary prime

$N(p_1)$ is an ordinary prime $\Rightarrow$ p_1 is a prime ideal.

Similarly, $\{3, 1-\sqrt{10}\}$ is a $\mathbb{Z}$-basis of p_2, $N(p_2) = \left| \begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} \right| = |-3| = 3$

so p_2 is a prime ideal.

$3\cdot 3 = 9 \in p_1 \cdot p_2$, $3(1-\sqrt{10}) - (1+\sqrt{10})\cdot 3 = 6 \in p_1 p_2$, so $9-6 = 3 \in p_1 p_2$

$\Rightarrow (3) \subseteq p_1 p_2 \Rightarrow p_1 \cdot p_2 \mid (3)$

$N(p_1 p_2) = N(p_1)\cdot N(p_2) = 3\cdot 3 = 9 = N((3))$

$p_1 p_2 \mid (3)$ and $N(p_1 p_2) = N((3)) \Rightarrow p_1 p_2 = (3)$

(or, to show $p_1 p_2 \mid (3)$, we could say $3 \in p_1$, so $p_1 \mid (3)$; $3 \in p_2$, so $p_2 \mid (3)$ $\Big\} \Rightarrow p_1 p_2 \mid (3)$ since p_1 and p_2 are prime ideals. (using unique factorization of ideals into product of prime ideals)

6) Second and direct solution:

As above $9 \in p_1 p_2$, $6 \in p_1 p_2$ so $9-6 = 3 \in p_1 p_2$, so $(3) \subseteq p_1 p_2$

To show $(3) = p_1 p_2$, it suffices to show $p_1 p_2 \subseteq (3)$:

$p_1 p_2 = (3, 1+\sqrt{10})(3, 1-\sqrt{10})$

$= (3\cdot 3, 3\cdot(1-\sqrt{10}), (1+\sqrt{10})\cdot 3, (1+\sqrt{10})(1-\sqrt{10}))$

$= (9, 3-3\sqrt{10}, 3+3\sqrt{10}, -9) \subseteq (3)$ since $9 = 3\cdot 3$

$3+3\sqrt{10} = 3(1+\sqrt{10})$

$3-3\sqrt{10} = 3(1-\sqrt{10})$

$-9 = 3(-3)$

since all of the 4 generators are in (3)