Math 366 - Quiz 1

Name and Student ID: SOLUTION

Question: Find all triples $(x, y, z) \in \mathbb{Z}^3$ such that $x^2 + y^2 = z^2$, $x > 0$, $y > 0$, $z > 0$ and $y + z = 245$.

$x^2+y^2=z^2 \Rightarrow (x,y,z) = (\pm k x_0, \mp k y_0, \mp k z_0)$ for some $k \in \mathbb{Z}^+$ where (x_0,y_0,z_0) is a primitive Pythagorean triple

$x>0, y>0, z>0 \Rightarrow (x,y,z) = (k x_0, k y_0, k z_0)$ for $k \in \mathbb{Z}^+$

where
$$\underbrace{\begin{aligned} x_0 &= 2st \\ y_0 &= s^2-t^2 \\ z_0 &= s^2+t^2 \end{aligned}}_{\text{Case 1}} \quad \text{OR} \quad \underbrace{\begin{aligned} x_0 &= s^2-t^2 \\ y_0 &= 2st \\ z_0 &= s^2+t^2 \end{aligned}}_{\text{Case 2}}$$
such that $\gcd(s,t)=1$, $s>t>0$, $s \not\equiv t \pmod 2$, $s,t \in \mathbb{Z}^+$

<u>In Case 1</u>: $y+z = k y_0 + k z_0 = k(s^2-t^2+s^2+t^2) = 2ks^2 = 245$ — no solution $s \in \mathbb{Z}$, $2 \nmid 245$

<u>In Case 2</u>, $y+z = k y_0 + k z_0 = 2kst + k(s^2+t^2) = k(s^2+2st+t^2) = 245$

$k \cdot (s+t)^2 = 5 \cdot 7^2$, $\left.\begin{aligned} s+t &\in \mathbb{Z}^+ \\ k &\in \mathbb{Z}^+ \end{aligned}\right\} \Rightarrow \boxed{k=5,\ s+t=7}$

$s+t=7$, $\gcd(s,t)=1$, $s>t>0$, $s \not\equiv t \pmod 2$,

Then $(s,t) = (6,1)$
$(5,2)$
$(4,3)$

$(s,t)=(6,1) \Rightarrow (x,y,z) = 5\cdot(6^2-1^2, 2\cdot6\cdot1, 6^2+1^2) = (175, 60, 185)$

$(s,t)=(5,2) \Rightarrow (x,y,z) = 5\cdot(25-4, 20, 25+4) = (105, 100, 145)$

$(s,t)=(4,3) \Rightarrow (x,y,z) = 5\cdot(16-9, 2\cdot4\cdot3, 16+9) = (35, 120, 125)$