# M E T U
# Department of Mathematics

| | Elementary Number Theory II<br>**FINAL** | |
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| Code : *Math 366*<br>Acad. Year : *2018-2019*<br>Semester : *Spring*<br>Instructor : *Tolga Karayayla*<br><br>Date : *25.05.2019*<br>Time : *9:30*<br>Duration : *120 minutes* | Last Name :<br>First Name : Student ID :<br>Department :<br>Signature :<br><br>**7** Questions on **4** Pages<br>SHOW DETAILED WORK! |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
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1. (8+12 pts.) a) Find two solutions of $x^2 - 18y^2 = 25$ in positive integers. (You can use $\sqrt{18} = [4, \overline{4, 8}]$).

b) What is the set of all solutions of $x^2 - 18y^2 = 25$?

2. (10 pts.) Solve the system of equations $x^2 + y^2 = z^2$, $x + y + z = 90$ in positive integers.

3. (5+10 pts.) a) Fill in the blanks with appropriate Gaussian integers (no explanation is necessary):
Let $\alpha \in \mathbb{Z}[i]$ be a Gaussian prime. If $N(\alpha)$ divides a power of 3, then $\alpha$ is an associate of ____.
If $N(\alpha)$ divides a power of 5, then $\alpha$ is an associate of either ____or ____.
If $N(\alpha)$ divides a power of 13, then $\alpha$ is an associate of either ____ or ____.
b) How many distinct Gaussian integers $\beta$ are there such that $N(\beta) = 3^2 \cdot 5 \cdot 13^2$? (Hint: Consider the factorization of $\beta$ as a product of Gaussian primes. What can you say about the norms of these prime factors?)

4. ( 10 pts.) Show that $I_{-21} = \mathbb{Z}[\sqrt{-21}]$ is not a UFD (Hint: Factorize 22 in $I_{-21}$).

5. ($3 \times 6$ pts.) a) Factorize the principal ideal (5) as a product of prime ideals in $I_{10} = \mathbb{Z}[\sqrt{10}]$.

b) Is the ideal $(5, \sqrt{10})$ a principal ideal in $I_{10}$?

c) Show that if $n$ is odd, then the equation $x^2 - 10y^2 = 5^n$ has no solution $(x, y) \in \mathbb{Z}^2$.

6. (12 pts.) Show that $I_{-19} = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a UFD (Hint: Use the theorem on Minkowski constants).

7. (8+7 pts.) a) Let $p \in \mathbb{Z}$ be a prime. Show that if $p$ does not divide $d$ and $d$ is a squarefree integer, then either the principal ideal $(p)$ is a prime ideal or $(p) = \alpha\beta$ for two (not necessarily distinct) prime ideals $\alpha$ and $\beta$ in the ring of integers $I_d$ of the quadratic extension $Q(\sqrt{d})$.

b) Assume $(p) = \alpha\beta$ as in part (a) above and suppose $\alpha \neq \beta$ in $I_d$. How many ideals $\delta$ are there in $I_d$ such that $N(\delta) = p^n$ where $n$ is a positive integer, and what are these ideals $\delta$?