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MATH 594 Theory of Special Functions

Homework 5

Part A (Pochhammer's Symbol)

Exercise 1 For non-negative integers n, m and $\alpha \in \mathbb{C}$, show that

(a) $(-m)_n = \begin{cases} 0 & , n > m \\ (-1)^n m!/(m-n)! & , n \leq m \end{cases}$

(b) $(\alpha)_{n+1} = (\alpha)_n(\alpha+n), \qquad (\alpha-1)(\alpha)_{n-1} = (\alpha-1)_n, \qquad (\alpha)_n = (-1)^n(1-n-\alpha)_n$

(c) $(\alpha)_{n-m} = (-1)^m \dfrac{(\alpha)_n}{(1-\alpha-n)_m}, \qquad (1-\alpha-n)_{n-m} = (-1)^{n-m}\dfrac{(\alpha)_n}{(\alpha)_m}, \qquad n \geq m$

(d) $\dbinom{m+2n}{m} = \dfrac{2^{2n}}{(2n)!}(\tfrac{1}{2}m+\tfrac{1}{2})_n\,(\tfrac{1}{2}m+1)_n, \qquad \dbinom{m+n}{m} = \dfrac{1}{n!}(m+1)_n$

(e) $\dbinom{\alpha}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!} & , \alpha = n \geq m \quad \text{(See definition!)} \\ \dfrac{(-1)^m(-\alpha)_m}{m!} & , \text{otherwise} \end{cases}$

Definition. (Binomial Coefficients): Recall that, when n and m are integers, binomial coefficients are defined by $\dbinom{n}{m} = C(n,m) = \dfrac{n!}{m!(n-m)!}$ in **combinatorial analysis**, and denotes the number of ways of choosing m objects from a collection of n distinct objects without regard to order.

Part B (Representation of elementary functions in terms of functions of the Hypergeometric type)

Exercise 2 Show that

(a) ${}_2F_1(\alpha,0;\gamma;z) = {}_1F_1(0;\gamma;z) = 1$

(b) ${}_2F_1(\alpha,\beta;\beta;z) = (1-z)^{-\alpha}\,{}_2F_1(\beta-\alpha,0;\beta;z) = (1-z)^{-\alpha}$

(c) ${}_1F_1(\alpha;\alpha;z) = \mathrm{e}^z\,{}_1F_1(0;\alpha;z) = \mathrm{e}^z$

(d) ${}_2F_1(\tfrac{1}{2}\alpha,\tfrac{1}{2}\alpha+\tfrac{1}{2};\tfrac{1}{2};z^2) = \tfrac{1}{2}\left[(1-z)^{-\alpha}+(1+z)^{-\alpha}\right]$

(e) ${}_2F_1(\tfrac{1}{2}\alpha+\tfrac{1}{2},\tfrac{1}{2}\alpha+1;\tfrac{3}{2};z^2) = \dfrac{1}{2\alpha z}\left[(1-z)^{-\alpha}-(1+z)^{-\alpha}\right]$

(f) ${}_2F_1(1,1;2;z) = -\dfrac{1}{z}\ln(1-z)$

(g) ${}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};z^2) = \dfrac{1}{2z}\ln\left(\dfrac{1+z}{1-z}\right)$

(h) ${}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2) = \dfrac{1}{z} \arcsin z$

(i) ${}_2F_1(\frac{1}{2}, 1; \frac{3}{2}; -z^2) = \dfrac{1}{z} \arctan z$

(j) ${}_1F_1(\alpha + 1; \alpha; z) = \left(1 + \dfrac{z}{\alpha}\right) \mathrm{e}^z$

Exercise 3 Show that the substitution $\xi = \sin^2 z$ transforms the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + n^2 y = 0$$

to an EHT. Then prove that

(a) $\cos(nz) = {}_2F_1(\frac{1}{2}n, -\frac{1}{2}n; \frac{1}{2}; \sin^2 z)$

(b) $\sin(nz) = n \sin z \, {}_2F_1(\frac{1}{2} - \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}n; \frac{3}{2}; \sin^2 z)$

if $-\frac{1}{2}\pi \leq z \leq \frac{1}{2}\pi$. Verify your results for the special case of $n = 2$.