EXERCISE SET #6

In the case of a degenerate or separable kernel $K(s,t)$ as

$$K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$$

where the functions $a_1(s), \ldots, a_n(s)$ and the functions $b_1(s), \ldots, b_n(s)$ are linearly independent, the Fredholm integral equation of the 2nd kind

$$g(s) = f(s) + \lambda \int K(s,t) g(t) dt$$

becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int b_i(t) g(t) dt \, .$$

The problem then reduces to finding the quantites c_i where

$$c_i = \int b_i(t) g(t) dt$$

such that

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} c_i a_i(s) \, .$$

Introducing the notation

$$f_i = \int b_i(t) f(t) dt \, , \qquad a_{ik} = \int b_i(t) a_k(t) dt \, ,$$

the problem further reduces to a system of n algebraic equations for the unknowns c_i,

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i \, .$$

The determinant $D(\lambda)$ of this system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \ldots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \ldots & -\lambda a_{2n} \\ \vdots & & & \\ -\lambda a_{n1} & -\lambda a_{n2} & \ldots & 1-\lambda a_{nn} \end{vmatrix},$$

which is s a polynomial in λ of degree at most n. Note that

(i) $D(0) = I_n$,

(ii) For all values of λ for which $D(\lambda) \neq 0$, the algebraic system, and thereby the integral equation, has a unique solution.

(iii) For all values of λ for which $D(\lambda) = 0$, the algebraic system, and the integral equation, either is insoluble or has an infinite number of solutions.

(iv) Setting $\lambda = 1/\mu$, we have the eigenvalue problem of matrix theory where they are given by the polynomial $D(\lambda) = 0$. They are also the eigenvalues of the integral equation with the corresponding solution $g(s)$ being the eigenfunction.

The inverse of the integral equation is defined by

$$g(s) = f(s) + \lambda \int \Gamma(s, t : \lambda) f(t) dt$$

where the function $\Gamma(s,t;\lambda)$ is the resolvent (or reciprocal) kernel that is given by

$$\Gamma(s,t;\lambda) = D(s,t;\lambda)/D(\lambda)$$

with

$$D(s,t;\lambda) = \begin{vmatrix} 0 & a_1(s) & a_2(s) & \dots & a_n(s) \\ b_1(t) & 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ b_2(t) & -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \vdots & & & & \\ b_n(t) & -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1-\lambda a_{nn} \end{vmatrix}.$$

Note that the roots of the equation $D(\lambda)=0$, i.e. the eigenvalues of the kernel $K(s,t)$, are the singular points of $\Gamma(s,t;\lambda)$.

<u>Fredholm Alternative Theorem</u>

(i) Either the nonhomogeneous Fredholm integral equation

$$g(s) = f(s) + \lambda\int K(s,t)g(t)dt$$

with a separable kernel and fixed λ possesses one and only one (unique) solution g(s) for arbitrary square-integrable functions $f(s)$ and $K(s,t)$, or

(ii) the nonhomogeneous integral equation has a solution if and only if the given function $f(s)$ satisfies the r orthogonality conditions

$$(f,\psi_{0i}) \equiv \int f(s)\psi_{0i}(s)ds = 0, \quad i = 1,2,\dots,r$$

where the solution is determined only up to an additive linear combination $\sum_{i=1}^{r} C_i g_{0i}$.

Here, g_{0i} stands for r linearly independent solutions of the homogeneous equation

$$g(s) = \lambda\int K(s,t)g(t)dt,$$

while ψ_{0i} stands for r linearly independent solutions of the transposed homogeneous equation

$$\psi(s) = \lambda\int K(t,s)\psi(t)dt.$$

Note that alternative (i) occurs for values of λ for which $D(\lambda) \neq 0$, in which case the homogeneous equation ($f=0$) has only the trivial solution $g=0$.

<u>Analogy of System of Linear Equations</u>

Consider a linear system of n equations in m unknowns represented in the matrix form

$$\underset{n\times m}{A}\;\underset{m\times 1}{X} = \underset{n\times 1}{B}$$

and consider the matrix A as composed of m column vectors $C_i \in \mathbf{R}^n$, $i = 1, 2, ..., m$, and n row vectors $R_j \in \mathbf{R}^m$, $j = 1, 2, ..., n$, that is,

$$A = \begin{bmatrix} C_1 & C_2 & \dots & C_m \\ \downarrow & \downarrow & \dots & \downarrow \\ \vdots & \vdots & & \vdots \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} R_1 & \rightarrow & \dots & \rightarrow \\ R_2 & \rightarrow & \dots & \rightarrow \\ \vdots & \vdots & & \vdots \\ R_n & \rightarrow & \dots & \rightarrow \end{bmatrix}.$$

Some notes:

(i) The column space of A, $\text{span}\{C_1, C_2, ..., C_m\} \subseteq \mathbf{R}^n$, and the row space of A, $\text{span}\{R_1, R_2, ..., R_n\} \subseteq \mathbf{R}^m$, have the same dimension, called rank of A, say r,

$$\dim\{\text{span}\{C_1, C_2, ..., C_m\}\} = \dim\{\text{span}\{R_1, R_2, ..., R_n\}\} = r$$

which is equal to the number of linearly independent column and row vectors, respectively.

(ii) Since column and row spaces of A are subspaces of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively, $r \le \min\{n, m\}$.

(iii) The solution space of the homogeneous system $AX = 0$ form a subpace of $\mathbf{R}^m$, called the kernel of A,

$$\ker\{A\} = \dim\{X \mid AX = 0\} = \dim\{\text{span}\{X_{01}, X_{02}, ..., X_{0p}\}\} \equiv p = m - r$$

which is the orthogonal complement to the row space of A in $\mathbf{R}^m$, because

$$AX = \begin{bmatrix} R_1 & \rightarrow & \dots & \rightarrow \\ R_2 & \rightarrow & \dots & \rightarrow \\ \vdots & \vdots & & \vdots \\ R_n & \rightarrow & \dots & \rightarrow \end{bmatrix} X = \begin{bmatrix} R_1 \cdot X \\ R_1 \cdot X \\ \vdots \\ R_1 \cdot X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(iv) The solution space to the transposed homogeneous system $A^T Y = 0$ form a subpace of $\mathbf{R}^n$, called the kernel of A^T,

$$\ker\{A^T\} = \dim\{Y \mid A^T Y = 0\} = \dim\{\text{span}\{Y_{01}, Y_{02}, ..., Y_{0q}\}\} \equiv q = n - r$$

which is the orthogonal complement to the column space of A in $\mathbf{R}^n$, because

$$A^T Y = \begin{bmatrix} C_1 & \rightarrow & \dots & \rightarrow \\ C_2 & \rightarrow & \dots & \rightarrow \\ \vdots & \vdots & & \vdots \\ C_m & \rightarrow & \dots & \rightarrow \end{bmatrix} Y = \begin{bmatrix} C_1 \cdot Y \\ C_2 \cdot Y \\ \vdots \\ C_m \cdot Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(v) (Fredholm Alternative) The nonhomogeneous system $AX = B$ has a solution if and only if B is in the column space of A, that is,

$$AX = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \\ \downarrow & \downarrow & \cdots & \downarrow \\ \vdots & \vdots & & \vdots \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix} X = x_1 C_1 + x_2 C_2 + \ldots + x_m C_m = B,$$

or it is in the orthogonal complement of the kernel of A^T, that is, B satisfies the $q = n - r$ constraints, $B \cdot Y_{0i} = 0$, $i = 1, 2, \ldots, q$.

1. Consider the integral equation

$$g(s) = f(s) + \lambda \int_0^1 K(s,t) g(t) dt$$

and show that for the kernels given below the system determinant $D(\lambda)$ has the given expression.

(a) $K(s,t) = \pm 1$ & $D(\lambda) = 1 \mp \lambda$.

(b) $K(s,t) = st$ & $D(\lambda) = 1 - \lambda/3$.

(c) $K(s,t) = s^2 + t^2$ & $D(\lambda) = 1 - 2\lambda/3 - 4\lambda^2/45$.

2. Invert the integral equation

$$g(s) = f(s) + \lambda \int_0^{2\pi} (\sin s \cos t) g(t) dt.$$

Ans.: $g(s) = f(s) + \int_0^{2\pi} \Gamma(s,t;\lambda) f(t) dt$ where $\Gamma(s,t;\lambda) = \lambda \sin s \cos t$.

3. Find the resolvent kernel for the integral equation

$$g(s) = f(s) + \lambda \int_{-1}^1 (st + s^2 t^2) g(t) dt.$$

Ans.: $\Gamma(s,t;\lambda) = \dfrac{st}{1 - 2\lambda/3} + \dfrac{s^2 t^2}{1 - 2\lambda/5}$.

4. Solve the homogeneous Fredholm integral equation

$$g(s) = \lambda \int_0^1 (e^s e^t) g(t) dt.$$

Ans.: Nontrivial solution $g(s) = 2ce^s/(e^2 - 1)$ for $\lambda = 2/(e^2 - 1)$.

5. Find the eigenvalues and eigenfunctions of the homogeneous integral equation

$$g(s) = \lambda \int_1^2 \left[st + (1/st) \right] g(t) dt.$$

Ans.: The eigenvalues $\lambda_{1,2} = \frac{1}{2}(17 \mp \sqrt{265})$.

6. Show that the integral equation

$$g(s) = f(s) + (1/\pi) \int_0^{2\pi} \sin(s+t) g(t) dt$$

possesses no solution for $f(s)=s$, but that it possesses infinitely many solutions when $f(s)=1$. Ans.: $g_0(s)=\psi_0(s)=\sin(s)+\cos(s)$ and for a solution $\int_0^{2\pi}\left(\sin(t)+\cos(t)\right)f(t)dt=0$.

7. Solve the integral equation

$$g(s)=f(s)+\lambda\int_0^{2\pi}\cos(s+t)g(t)dt$$

and find the condition that $f(s)$ must satisfy in order that this equation has a solution when λ is an eigenvalue. Obtain the general solution if $f(s)=\sin s$, considering all possible cases. Ans.: $\lambda_1=1/\pi$ and $\int_0^{2\pi}\cos(t)f(t)dt=0$ & $\lambda_1=-1/\pi$ and $\int_0^{2\pi}\sin(t)f(t)dt=0$. For $\lambda_1=1/\pi$ and $f(s)=\sin s$, we have $g(s)=\frac{1}{2}\sin s+c\cos(s)$ where c is an arbitrary constant.

8. Show that the integral equation

$$g(s)=\lambda\int_0^{\pi}(\sin s\sin 2t)g(t)dt$$

has no eigenvalues. Ans.: $D(\lambda)=1$.

9. Solve the integral equation

$$g(s)=1+\lambda\int_{-\pi}^{\pi}e^{i\omega(s-t)}g(t)dt$$

considering separately all the exceptional cases.

Ans.: For $\lambda\neq 1/2\pi$: $g(s)=1+\frac{2\pi\lambda}{1-2\pi\lambda}$ when $\omega=0$; $g(s)=1$ when $0\neq\omega\in\mathbb{Z}$; $g(s)=1+\frac{2\lambda\sin(\omega\pi)}{\omega(1-2\pi\lambda)}e^{i\omega s}$ when $\omega\notin\mathbb{Z}$. For $\lambda=1/2\pi$: $g(s)=1+\frac{ce^{i\omega s}}{2\pi}$ when $0\neq\omega\in\mathbb{Z}$; no solution when $\omega=0$ and $\omega\notin\mathbb{Z}$.

10. In the integral equation

$$g(s)=s^2+\lambda\int_0^1\sin(st)g(t)dt$$

replace $\sin(st)$ by the first two terms of its power-series development

$$\sin(st)=st-\tfrac{1}{3!}(st)^3+\ldots$$

and obtain an approximate solution.

11. In the integral equation

$$g(s)=e^s-s-\int_0^1 s(e^{st}-1)g(t)dt$$

replace the kernel by the first three terms of its power-series development

$$K(s,t)=s(e^{st}-1)\approx s^2t+\tfrac{1}{2}s^3t^2+\tfrac{1}{6}s^4t^3$$

and obtain an approximate solution.