POISSON'S SUM FORMULA

Let us consider a periodic impulse train

$$s(t) = \sum_n \delta(t - nT) .$$

Because *s(t)* is periodic with period *T*, it can be expanded into a Fourier series:

$$s(t) = \sum_n c_n e^{\frac{j2\pi n}{T}t}$$

The FS coefficients are obtained as follows:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} s(t)e^{-\frac{j2\pi n}{T}t}dt = \frac{1}{T}\int_{-T/2}^{T/2}\sum_k \delta(t-kT)e^{-\frac{j2\pi n}{T}t}dt = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-\frac{j2\pi n}{T}t}dt = \frac{1}{T}$$

Therefore

$$s(t) = \frac{1}{T}\sum_n e^{\frac{j2\pi n}{T}t} .$$

Or

$$\boxed{\sum_n \delta(t - nT) = \frac{1}{T}\sum_n e^{\frac{j2\pi n}{T}t}}$$

The Fourier transform of *s(t)* is found easily:

$$\boxed{s(t) = \sum_n \delta(t - nT) \quad \leftrightarrow \quad S(f) = \frac{1}{T}\sum_n \delta\left(f - \frac{n}{T}\right)}$$

where we used $e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0)$.

Now, let's take an arbitrary energy signal *g(t)* and convolve it by *s(t):*

$$x(t) = g(t) \otimes s(t) = \sum_n g(t - nT)$$

Find the FT:

$$X(f) = G(f)S(f)$$

Using the sampling property of impulses, we have

$$\sum_n g(t - nT) \quad \leftrightarrow \quad \frac{1}{T}\sum_n G\left(\frac{n}{T}\right)\delta\left(f - \frac{n}{T}\right)$$

Finally by finding the inverse FT of the RHS of above relation, we obtain

$$\boxed{\sum_n g(t-nT) = \frac{1}{T}\sum_n G\left(\frac{n}{T}\right)e^{j\frac{2\pi n}{T}t}}$$

This relation is known in the literature as the Poisson's Sum Formula. Note that for the special case of $g(t)=\delta(t)$, $G(f)=1$ we obtain the formula concerning the impulse train.