A Proof of Neyman-Pearson Lemma

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In this note a proof of Neyman-Pearson Lemma is provided, which is a slightly modified version of the one in Van Trees' book[1].

We consider a simple binary hypothesis testing problem.

Consider an observation **r** which is a real vector in observation space $\boldsymbol{Z}$**.** The pdf's of **r** under both hypotheses, $p(\boldsymbol{r}|H_0)$ and $p(\boldsymbol{r}|H_1)$ are known.

We want to find the region $\boldsymbol{Z}_1$ where we decide on H_1 so that $P_D = \int_{\boldsymbol{Z}_1} p(\boldsymbol{r}|H_1)d\boldsymbol{r}$ is maximum while $P_F = \int_{\boldsymbol{Z}_1} p(\boldsymbol{r}|H_0)d\boldsymbol{r} = \alpha$, where $0 < \alpha < 1$. We assume that $p(\boldsymbol{r}|H_0)$ is bounded (no impulses) so that the constraint on P_F is given as an equality.

Define the objective function including the Lagrange multiplier as

$$F = P_D - \lambda(P_F - \alpha) = \int_{\boldsymbol{Z}_1}[p(\mathbf{r}|H_1) - \lambda p(\mathbf{r}|H_0)]d\mathbf{r} + \lambda\alpha.$$

For any given value λ the region $\boldsymbol{Z}_1$that maximizes F and hence P_D, under the constraint $P_F = \alpha$ is clearly given by

$$\boldsymbol{Z}_\mathbf{1} = \{\mathbf{r} \in Z | p(\mathbf{r}|H_1) > \lambda\, p(\mathbf{r}|H_0)\}.$$

This result directly yields the likelihood ratio test:

$$\Lambda(\mathbf{r}) = \frac{p(\mathbf{r}|H_1)}{p(\mathbf{r}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,.$$

But, what is λ ? We find it from the constraint

$$P_F = \int_{\boldsymbol{Z}_\mathbf{1}(\lambda)} p(\boldsymbol{r}|H_0)d\boldsymbol{r} = \alpha\,,$$

or employing Lebesgue integration we have

$$\int_\lambda^\infty p_{\Lambda|H_0}(\Lambda|H_0)d\Lambda = \alpha.$$

This integral equation is solved to obtain the required threshold λ. We note that in many problems the likelihood ratio can be reduced to a much simpler *sufficient statistic* and, instead of obtaining $p_{\Lambda|H_0}$ explicitly and solving the integral equation, an equivalent problem is solved in terms of the sufficient statistic to get the test.

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[1] Harry L. Van Trees, *Detection, Estimation and Modulation Theory*, Part 1, John Wiley and Sons, 1968.