p-POWER POINTS AND MODULES OF CONSTANT p-POWER JORDAN TYPE

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Abstract. We study finitely generated modules over $k[G]$ for a finite abelian p-group G with p dividing the $char(k)$ is p, through restrictions to certain subalgebras of $k[G]$. Mail we obtained the following:

We define p-power points, shifted cyclic p-power order subgroups of $k[G]$ and give characterizations of these. We define modules of constant p^t-Jordan type, constant p^t-power-Jordan type as generalizations of modules of constant Jordan type, and p^t-support, non-maximal p^t-support spaces. We obtain a filtration of modules of constant Jordan type with modules of constant p-power Jordan type as the last term and give examples of non-isomorphic modules of constant p-power Jordan type having the same constant Jordan type.

1. Introduction

Let G be a finite group, k be an algebraically closed field of characteristic $p > 0$, and M be a finitely generated module over the group algebra $k[G]$. The restriction of M to a subalgebra $k[H]$ of $k[G]$ is denoted by $M{\downarrow_H}$ where H is a subgroup of the group of units of $k[G]$. Recall that for a p-group G, a projective $k[G]$-module is a free $k[G]$-module. The following two theorems from late 70's show that studying modules via restrictions is a powerful tool.

Chouinard's Theorem [**Ch**]. Let G be a finite group. Then M is a projective $k[G]$-module if and only if $M{\downarrow_E}$ is a projective $k[E]$-module for every elementary abelian p-subgroup E of G.

Dade's Lemma [**Da**, Lemma 11.8]. Let $E = \langle e_1, \ldots, e_n\rangle$ be an elementary abelian p-group of order p^n and $x_\alpha = \alpha_1(e_1 - 1) + \cdots + \alpha_n(e_n - 1)$. Then M is a projective $k[E]$-module if and only if $M{\downarrow_{\langle 1+x_\alpha\rangle}}$ is a projective $k[\langle 1 + x_\alpha\rangle]$-module for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$.

Dade's Lemma initiated further study of modules via restrictions to certain subgroups of the unit group of $k[E]$. It is one of the foundations of the rank variety $V_E^{\mathbf{r}}(M)$ of a $k[E]$-module M for an elementary abelian p-group E defined by Carlson

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[**Ca**] as the set

$$V_E^{\mathbf{r}}(M) = \{0\} \cup \{\alpha \in k^n \mid M{\downarrow}_{\langle 1+x_\alpha\rangle} \text{ is not free}\}.$$

The group $\langle 1+x_\alpha\rangle$ appearing in Dade's Lemma and also in the definition of rank variety is called a shifted cyclic subgroup of $k[E]$ in [**Ca**]. For an abelian p-group G, we define a *shifted (cyclic p-power) subgroup* H of $k[G]$ as a (cyclic, p-power order) subgroup of the unit group of $k[G]$ provided that $k[G]$ is free as a $k[H]$-module.

The rank variety is generalized to restricted Lie algebras [**FPa**], [**FPa1**]; infinitesimal groups [**SFB**], [**SFB1**]; finite group schemes [**FP**]. In [**FP**], shifted cyclic subgroups are generalized to p-points for group schemes. When G is a finite abelian p-group, a p-point is the same as a flat point defined in [**Fa**]. In a subsequent recent work, modules of constant Jordan type for finite group schemes are introduced by Carlson, Friedlander and Pevtsova [**CFP**]. The common theme in all of these is that a $k[G]$-module is studied via the Krull-Schmidt decomposition, equivalently the Jordan block decomposition, of the restrictions of the module to subalgebras isomorphic to the group algebra of a cyclic group of order p, namely $k[X]/X^p$. In fact, for a module the set of Jordan types is first defined in [**Oz**] as a local invariant consisting of the *multiplicities* of the Jordan blocks in the Jordan canonical form of the matrix of the module at shifted cyclic subgroups of abelian p-groups of small rank with a particular emphasis on the group $C_2 \times C_4$, and its Carlson modules L_ζ's. Using Jordan decompositions at shifted cylic subgroups referred as the *set of multiplicities*, it is shown in [**Ka**] that certain type of $k[C_2{\times}C_4]$-modules having the same k-dimension and variety, such as $(L_{\zeta_\alpha^n})^m$ and $(L_{\zeta_\alpha^m})^n$ for ζ_α in the polynomial part of $H^2(C_2{\times}C_4;k)$, can be distinguished. Moreover, for a $k[C_2{\times}C_4]$-module and a Moore space X realizing M, a geometric interpretation of the multiplicities in the Jordan decompostion of $M{\downarrow}_C$ in terms of the Betti numbers of the fixed point set X^C for a cyclic subgroup C is given in [**Ka2**].

In this article, we study $k[G]$-modules via the restrictions to subalgebras isomorphic to $k[X]/X^{p^t}$, for $t = 1, \dots, m$, instead of only $t = 1$, for a finite abelian p-group G of exponent p^m. We generalize the notion of p-points of [**FP**] to *p-power points of degree t* (or simply *p^t-points*) in 3.4. As a natural consequence we define *modules of constant p^t-Jordan type, modules of constant p^t-power Jordan type*, see 4.1, and denote the sets of these modules by $C^t(G)$, $\mathcal{C}^t(G)$ respectively. We refer to modules of constant p^m-power Jordan type simply as *modules of constant p-power Jordan type*. We obtain a decreasing filtration of modules of constant Jordan type having the set of modules of constant p-power Jordan type as the last term:

$$\mathcal{C}_G^1 \supseteq \mathcal{C}_G^2 \supseteq \cdots \supseteq \mathcal{C}_G^m.$$

Endotrivial $k[G]$-modules, $k[G]$-modules with equal image property are examples of constant p-power Jordan type modules as disscussed in Section 4. For an example of a module in $\mathcal{C}_G^t$ but not in $\mathcal{C}_G^{t+1}$ see 4.4. Since changing the algebra $k[X]/X^p$ to $k[X]/X^{p^t}$ makes no change in the arguments showing that modules of constant Jordan type are closed under taking direct sums and tensor products in [**CFP**], we also have that C_G^t, and hence $\mathcal{C}_G^t$ are closed under direct sums and tensor products. Our approach of considering p-power points allows us to distinguish modules that are not distinguishable by considering only p-points as demonstrated in Examples 4.9–4.11. We generalize some results/definitions of [**Ca**], [**FP**], [**FPS**], [**Ka**].

We need to introduce some notation before stating our theorems. Let J_G (or simply J when there is no ambiguity) denote the Jacobson radical of $k[G]$. It is known that for an elementary abelian p-group E, if x is a non-zero element of J_E^2 then $k[E]$ is not free as a $k[\langle 1+x\rangle]$-module [**Ca**, Lemma 6.1]. However, when the group G is not elementary abelian, $k[G]$ is certainly free as a $k[\langle g^p\rangle]$-module for any g in G of order p^2 or greater and $g^p-1=(g-1)^p$ is in J_G^2. This leads us to define the *pseudo-radical-square* $\mathbb{J}^{(2)}$ as a substitute for J_E^2 so that $g^p-1 \notin \mathbb{J}^{(2)}$; see 2.5. We write $J = L \oplus \mathbb{J}^{(2)}$ and $x = x_L + x_{\mathbb{J}^{(2)}}$ for $x \in J$ with $x_L \in L$, $x_{\mathbb{J}^{(2)}} \in \mathbb{J}^{(2)}$. Although $\mathbb{J}^{(2)}$ is not necessarily an ideal, $\mathbb{J}^{(2)}$ and its vector space complement L in J are closed under the Frobenius homomorphism F given by $F(a) = a^p$. Our main theorems, Theorem 3.2 and Theorem 3.5, are generalizations of Lemma 6.1 and 6.4 in [**Ca**] and also those of Theorem 4.1 and Theorem 4.3 in [**Ka**] respectively. Theorem 3.2 gives a characterization of p-power points and shifted cyclic subgroups of $k[G]$.

Theorem 3.2. *Suppose that G is an abelian p-group and $0 \neq x \in J_G$. Then $k[G]$ is free as a $k[\langle 1+x\rangle]$-module if and only if $x^{|\langle 1+x\rangle|/p} \notin \mathbb{J}^{(2)}$ if and only if $|\langle 1+x_L\rangle| = |\langle 1+x\rangle|$.*

It is not difficult to find examples of $k[G]$-modules M, such that for x and y in $J_G \backslash J_G^2$ with $x \equiv y \pmod{J_G^2}$, the direct sum decompositions, i.e., the Jordan types, of $M{\downarrow}_{\langle 1+x\rangle}$ and $M{\downarrow}_{\langle 1+y\rangle}$ are not the same. This phenomenon does not occur when $G = C_2 \times C_2$ and x is a 2-point, or when $G = C_2 \times C_4$ and x is a 4-point [**Ka**], i.e., the Jordan type at x is independent of $x_{\mathbb{J}^{(2)}}$. In general, for the elements x, y of

$P^t(G)$ with $x \equiv y \mod (\mathbb{J}^{(2)})$, we show by Theorem 3.5 that for a $k[G]$-module of k-dimension divisible by p^t, the Jordan block decomposition of the matrix of x on M is of maximal possible Jordan type if and only if the Jordan block decompositon of the matrix of y on M is of maximal possible Jordan type.

Theorem 3.5. *Let x, y be in $P^t(G)$ and M be a finitely generated $k[G]$-module. If $x \equiv y \mod (\mathbb{J}^{(2)})$, then $M{\downarrow_{\langle 1+x\rangle}}$ is free if and only if $M{\downarrow_{\langle 1+y\rangle}}$ is free.*

By Theorem 3.5 we are able to define an equivalence relation on $P^t(G)$ by setting $x \sim y$ if and only if $M{\downarrow_{\langle 1+x\rangle}}$ is free if and only if $M{\downarrow_{\langle 1+y\rangle}}$ is free for every finitely generated $k[G]$-module M. We denote the quotient set of this relation by $\mathcal{P}^t(G)$. Thus, by Theorem 3.5 there are surjective maps

$$J/\mathbb{J}^{(2)} \longrightarrow \mathcal{P}^t(G) \quad \text{and} \quad P^t(G)/{\mathbb{J}^{(2)}}^t \longrightarrow \mathcal{P}^t(G).$$

When G is elementary abelian the fist map is shown to be a bijection by Carlson [**Ca**], however when the group is non-elementary abelian that map has a non-trivial kernel, see 3.7. Also, for $x \in J$ it is immediate that x is a p^t-point if and only if $x^{p^{t-i}}$ is a p^i-point, and there is a map $F^i : \mathcal{P}^t(G) \longrightarrow \mathcal{P}^{t-i}(G)$ for $i = 1, \ldots, m$, where F is the Frobenius homomorphism. Obviously there is a one to one correspondence of $\mathcal{P}^1(G)$ with $P(G)$ as well as of $P(G)$ with $\mathrm{Proj}|G|$, where $|G|$ is the cohomological support variety of G; see [**FP**], [**Ca**], [**AS**], [**Qu**]. Thus there are maps $J/\mathbb{J}^{(2)} \longrightarrow \mathcal{P}^t(G) \longrightarrow \mathcal{P}^1(G) \longrightarrow \mathrm{Proj}|G|$. At this point for each t, $t = 1 \ldots, m$, it is natural to define the *p^t-support space* $\mathcal{P}^t(G)_M$, and the *p^t-non-maximal support space* $\Gamma^t(G)_M$ for any finitely generated $k[G]$-module M as follows:

$\mathcal{P}^t(G)_M = \{\, [x] \in \mathcal{P}^t(G) \mid M{\downarrow_{\langle 1+x\rangle}}$ is not free$\,\}$ and
$\Gamma^t(G)_M = \{\, [x] \in \mathcal{P}^t(G) \mid M{\downarrow_{\langle 1+y\rangle}}$ is not of maximal Jordan type for some representative y of $[x]\,\}$

with the partial order on Jordan types given by the usual dominance order as in [**FPS**]. Since $M{\downarrow_{\langle 1+x\rangle}}$ is not free if and only if $\dim_k(xM) \neq \frac{\dim_k(M)}{|\langle 1+x\rangle|}(|\langle 1+x\rangle| - 1)$, $\mathcal{P}^t(G)_M$ generalizes Carlson's rank variety from modules over elementary abelian p-groups to any abelian p-group. Changing the algebra $k[X]/X^p$ to $k[X]/X^{p^t}$ makes no difference in the arguments for $t = 1$ in [**CFP**], hence we deduce that $M{\downarrow_{\langle 1+y\rangle}}$ has the same Jordan type as $M{\downarrow_{\langle 1+x\rangle}}$ provided that x, y are equivalent p^t-points and $M{\downarrow_{\langle 1+x\rangle}}$ has maximal Jordan type, for each $t = 1, \ldots, m$. Thus $\Gamma^t(G)_M$ is well-defined. Moreover, we have the analogous set inclusions

$$\Gamma^t(G)_M \subseteq \mathcal{P}^t(G)_M \subseteq \mathcal{P}^t(G),$$

where the first inclusion is equality if and only if the second set inclusion is proper. Furthermore, $\Gamma^t(G)_M = \emptyset$ if and only if M is of constant p^t-Jordan type, i.e., M is in C^t_G, and $\Gamma^1(G)_M = \cdots = \Gamma^t(G)_M = \emptyset$ if and only if M is of constant p^t-power Jordan type, i.e., M is in $\mathcal{C}^t_G$. Taking direct sum commutes with restriction, hence we have

$$\mathcal{P}^t(G)_{M\oplus N} = \mathcal{P}^t(G)_M \cup \mathcal{P}^t(G)_N.$$

Since freeness at a p^t-point x is determined by the p-point $x^{p^{(t-1)}}$, by Lemma 3.9 in [**FP**] we have

$$\mathcal{P}^t(G)_{M\otimes N} = \mathcal{P}^t(G)_M \cap \mathcal{P}^t(G)_N.$$

We expect also that

$$\Gamma^t(G)_{M\oplus N} = \Gamma^t(G)_M \cup \Gamma^t(G)_N,$$

and since G has a unique maximal elementary abelian subgroup we expect that

$$\Gamma^t(G)_{M\otimes N} = (\Gamma^t(G)_M \cup \Gamma^t(G)_N) \cap (\mathcal{P}^t(G)_M \cap \mathcal{P}^t(G)_N).$$

In the $t = 1$ case, $\mathcal{P}^t(G)$, $\Gamma^t(G)$ are well-studied and they have further structures. For instance, $P(G)$ of [**FP**] is a variety known as the support variety, its subvarieties $P(G)_M$ are the closed sets of the Zariski topology on $P(G)$ for M a $k[G]$-module, $\Gamma(G)_M$ is a closed subset of $P(G)_M$, and there is a scheme structure on the pojectivized cohomological support variety $Proj|G|$; see [**Ca**], [**FP**], [**FPS**] *et al.* The analogous properties remain to be explored in our setting for $t > 1$, as well as the above-mentioned filtration of modules of constant Jordan type.

The outline of the article is as follows. Preliminary results are in the second section, main theorems are proved in the third section, modules of constant p-power Jordan type and the main examples of the article constitute the fourth section.

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2. Preliminary Results

We first include two well known results from the literature, then include two lemmas which are given in [**Ca**] when the group is an elementary abelian p-group. Then we define the pseudo-radical-square $\mathbb{J}^{(2)}$ of $k[G]$ and prove a lemma involving the elements of $\mathbb{J}^{(2)}$. After that we give our preliminary results that are used in the next section.

Let H be a finite group, the element $\nu_H := \sum_{h\in H} h$ of $k[H]$ is referred as the *norm* element of the group algebra $k[H]$. When P is a finite p-group the group algebra $k[P]$ is a local ring with the unique maximal ideal J_P and the unique minimal left ideal $k\nu_P$. Also the notions of freeness, projectivity, and injectivity coincide for $k[P]$-modules. In the following lemma we list several equivalent conditions for determining the freeness of a $k[P]$-module.

Lemma 2.1. *Let P be a finite p-group, and M be a finitely generated $k[P]$-module. Then $\dim_k(\nu_P M) \leq \frac{1}{|P|}\dim_k(M)$. Moreover, the following are equivalent.*

(i) $\dim_k(\nu_P M) = \frac{\dim_k(M)}{|P|}$.
(ii) *M is a free $k[P]$-module.*

In particular, if $P = \langle g\rangle$ is of order p^t, then the following are equivalent.

(iii) $\dim_k((g-1)^{p^t-1}M) = \frac{\dim_k(M)}{p^t}$.
(iv) $\dim_k((g-1)M) = (p^t-1)\frac{\dim_k(M)}{p^t}$.
(v) $\dim_k((g^{p^{(t-1)}}-1)^{p-1}M) = \frac{\dim_k(M)}{p}$.
(vi) $\ker(g-1 \text{ on } M) = (g-1)^{p^t-1}M$.
(vii) $\ker((g-1)^{p^t-1} \text{ on } M) = (g-1)M$.

Proof. The first part follows from Lemma 5.10.2 in [**Be**] as $\dim_k(\nu_P M)$ is the number of free summands in the decomposition of M. The second part follows from the fist part using the fact that for $P = \langle g\rangle$ of order p^t, $\nu_P = (g-1)^{p^t-1}$ and $\nu_{\langle g^{p^{(t-1)}}\rangle} = (g^{p^{(t-1)}}-1)^{p-1} = (g-1)^{p^t-p^{(t-1)}}$. □

The following lemma is used in the proof of Theorem 3.2, it is mod-p binomial theorem as it is well known that the binomial coefficients satisfy the congruence $\binom{p^n-1}{j} \equiv (-1)^j$.

Lemma 2.2. *Let a, b be elements of a commutative k-algebra with $char(k) = p > 0$, and m be a positive integer. Then*

$$(a+b)^{p^m-1} = a^{p^m-1} - a^{p^m-2}b + \cdots - ab^{p^m-2} + b^{p^m-1}.$$

Some useful properties of the elements of the Jacobson radical J_G of $k[G]$ are summarized in the following lemma. They can be seen easily from the mod-p binomial theorem as the binomial coefficient $\binom{p^n}{j} \equiv 0$ for $j \leq p^n-1$.

Lemma 2.3. *Let x be a non-zero element of J_G.*

(i) $|\langle 1+x\rangle| = p^t$ *for some $t = 1,\ldots,m$ (recall that the exponent of G is p^m).*

(ii) $\langle 1+x\rangle = C_{p^t}$ *if and only if* $x^{p^t}=0$ *and* $x^{p^{(t-1)}}\neq 0$.

(iii) $|\langle 1+x^{p^l}\rangle| = p$ *if and only if* $x^{p^{l+1}}=0$ *and* $x^{p^l}\neq 0$.

(iv) $\dim_k(k[\langle 1+x\rangle]) = p^t$ *if and only if* $x^{p^t}=0$ *and* $x^{p^t-1}\neq 0$.

(v) $\dim_k(k[\langle 1+x^{p^{(t-1)}}\rangle]) = p$ *if and only if* $x^{p^t}=0$ *and* $x^{p^t-p^{(t-1)}}\neq 0$.

(vi) *If* $k[\langle 1+x\rangle]\cong k[C_{p^t}]$ *then* $k[\langle 1+x^{p^{(t-1)}}\rangle]\cong k[C_p]$.

Note that the converse of the statement (vi) of Lemma 2.3 is not true true in general, see Example 2.7 (1), (2).

Notation 2.4. Let $G=\langle g_1,\dots,g_r\rangle$ be a an abelian p-group of p-rank r and of exponent p^m with $\langle g_i\rangle\cong C_{p^{n_i}}$ for $i=1,\dots,r$ with $n_1\leq\cdots\leq n_r=m$. The unique maximal elementary abelian p-subgroup E of G can be written as $E=\langle e_1,\dots,e_r\rangle$ with $e_i=g_i^{p^{(n_i-1)}}$ for $i=1,\dots,r$. Define the *shifted basis* for $k[G]$ as the set

$$\mathcal{B}=\{\,(g_1-1)^{i_1}\cdots(g_r-1)^{i_r}\,|\; i_j=0,1,\dots,p^{n_j}-1,\quad j=1,\dots,r\}$$

together with the lexicographic order. In addition, taking $(g_i-1)^{p^{n_i-t}}=0$, for $t>n_i$ and $i=1,\dots,r$, we define the subspaces

$$L^t=k(g_1-1)^{p^{n_1-t}}\oplus\cdots\oplus k(g_r-1)^{p^{n_r-t}} \text{ and } N^t=L^1\oplus\cdots\oplus L^t\backslash L^1\oplus\cdots\oplus L^{t-1}$$

for $t=1,\dots,m$. Also we set $L=L^1\oplus\cdots\oplus L^m$.

Note that $L^t=\ker\{F^t: J\longrightarrow J\}\cap\{(g_i-1)^{p^{i_j}}\,|\,i=1,\dots,r,\;\; i_j=1,\dots,n_i-1\}$.

We write $x=x_U+x_V$ with x_U in U and x_V in V when x is in a subspace $U\oplus V$.

Definition 2.5. The *pseudo-radical-square* $\mathbb{J}^{(2)}$ of $k[G]$ is the k-subspace of J_G^2 spanned by the elements in the set

$$\mathcal{B}\backslash(\{1\}\cup\{\,(g_1-1)^{p^{i_1}},\dots,(g_r-1)^{p^{i_r}}\,|\; i_j=0,1,\dots,n_i-1,\quad j=1,\dots,r\}).$$

Let ${\mathbb{J}^{(2)}}^t$ denote the elements of $\mathbb{J}^{(2)}$ which have nilpotency at most p^t, i.e.,

$${\mathbb{J}^{(2)}}^t=\ker\{F^t:\mathbb{J}^{(2)}\longrightarrow\mathbb{J}^{(2)}\}$$

and

$$J_G=L\oplus\mathbb{J}^{(2)}.$$

Remark 2.6. Since $(g_i-1)^{p^t}\notin J_G\cdot J_E$, for any $t=1,\dots,n_i-1$, we have the following inclusions

$$J_G^2\supseteq\mathbb{J}^{(2)}\supseteq J_G\cdot J_E\supseteq J_E^2\ ,$$

which all become equalities when the group G is elementary abelian. Although the middle inclusion is equality for the non-elementary abelian 2-group $G=C_2\times C_4$, it is a proper inclusion in most cases; even for $G = \langle g \rangle \cong C_9$, we have $(g-1)^2$ in $\mathbb{J}^{(2)} \backslash J_G \cdot J_E$. It should be noted that

$$J_G/\mathbb{J}^{(2)} \supseteq J_E/J_E^2 \qquad \text{and} \qquad J_G/\mathbb{J}^{(2)} \cong L \cong \oplus_{i=1}^{r} \oplus_{i_j=0}^{n_i-1} J_{\langle g_i^{p^{i_j}} \rangle}/J^2_{\langle g_i^{p^{i_j}} \rangle}.$$

In the particular cases, namely, $p = 2$ and $m \leq 2$ the space $\mathbb{J}^{(2)}$ is an ideal of $k[G]$ even though it is not even a subring of $k[G]$ in general. For some results involving the elements of $\mathbb{J}^{(2)}$ see Proposition 3.1, Theorem 3.2, Lemma 2.8, Lemma 2.10.

The following example provides counterexamples to several expectations. Namely, parts (1), and (2) provide examples showing that the converse of Lemma 2.3 (vi) is not true. Part (4) provides a counterexample to the converse of Proposition 3.1 and to that of Theorem 3.2. Parts (1), (2), (3) provide examples of x with $|\langle 1+x \rangle| \cong C_{p^t}$ but $k[\langle 1+x \rangle] \ncong k[C_{p^t}]$.

Example 2.7. Let $G = \langle g, h \rangle$ be an abelian 3-group where g, h are of orders 3, 27 respectively. In each case $x \in J_G$.

(1) Let $x = (g-1)^2 + (h-1)^4$. Then $x \in \mathbb{J}^{(2)}$, $x^3 \in \mathbb{J}^{(2)}$ and $\langle 1+x^3 \rangle \cong C_3$, $\langle 1+x \rangle \cong C_9$. Moreover $k[\langle 1+x^3 \rangle] \cong k[C_3]$ but $k[\langle 1+x \rangle] \ncong k[C_9]$.

(2) Let $x = (h-1)^4 + (h-1)^9$. Then $x \notin \mathbb{J}^{(2)}$ but $x^3 \in \mathbb{J}^{(2)}$, and $\langle 1+x^3 \rangle \cong C_3$, $\langle 1+x \rangle \cong C_9$. Moreover $k[\langle 1+x^3 \rangle] \cong k[C_3]$ but $k[\langle 1+x \rangle] \ncong k[C_9]$.

(3) Let $x = (h-1)^2 + (h-1)^7$. Then $x \in \mathbb{J}^{(2)}$, however $x^2 = (h-1)^4 + 2(h-1)^9 + (h-1)^{14} \notin \mathbb{J}^{(2)}$, $x^3 = (h-1)^6 \in \mathbb{J}^{(2)}$, and $x^9 = (h-1)^{18} \in \mathbb{J}^{(2)}$. We have $\langle 1+x^9 \rangle \cong C_3$, and $\langle 1+x \rangle \cong C_{27}$, but $k[\langle 1+x^9 \rangle] \ncong k[C_3]$, $k[\langle 1+x \rangle] \ncong k[C_{27}]$.

(4) Let $x = (g-1) + (h-1)^4$. Then $x \notin \mathbb{J}^{(2)}$, $x^3 = (h-1)^{12} \in \mathbb{J}^{(2)}$, $x^6 \neq 0$, $x^8 \neq 0$ and $\langle 1+x \rangle \cong C_9$. Hence $k[\langle 1+x \rangle] \cong k[C_9]$ and $k[\langle 1+x^3 \rangle] \cong k[C_3]$.

The following lemma is an easy consequence of the definitions of L, $\mathbb{J}^{(2)}$ and the properties of elements of J_G given in Lemma 2.3.

Lemma 2.8. *Suppose that $0 \neq x \in J_G = L \oplus \mathbb{J}^{(2)}$.*

(i) *If $x \in \mathbb{J}^{(2)}$ then $x^{p^i} \in \mathbb{J}^{(2)}$ for $i \geq 0$, similarly if $x \in L$ then $x^{p^i} \in L$.*

(ii) *Suppose that $x = x_L + x_{\mathbb{J}^{(2)}}$ and $|\langle 1+x \rangle| = p^t$. Then $x^{p^{t-1}} \notin \mathbb{J}^{(2)}$ if and only if $|\langle 1+x_L \rangle| = |\langle 1+x \rangle|$.*

Next we prove a crutial technical lemma to be used in the proofs of Proposition 3.1 and Theorem 3.2. Setting up some notation to simplify its presentation is useful.

Notation 2.9. In order to simplify the notation in Lemma 2.10 we define in J_G the elements

$$\mu_j = \prod_{i=1, i\neq j}^{r} (e_i - 1) \qquad \text{and} \qquad \mu = (e_j - 1)\mu_j.$$

Note that μ and μ_j are in J_E as well, moreover $\nu_E = \sum_{g\in E} g = \mu^{p-1}$; recall that E is the unique maximal elementary abelian p-subgroup of G.

Lemma 2.10. *Suppose that $z \in \mathbb{J}^{(2)}$, $z^{p^t} = 0$, and $w = z^{p^{(t-1)}} \neq 0$. Then the following equalities hold for the elementary abelian subgroup $A = \langle e_1, \ldots, e_r, 1+w\rangle$ of the units of $k[G]$, and $u_\alpha = 1+\sum_{i=1}^r \alpha_i(e_i-1)+\alpha_{r+1}w$ where $\alpha = (\alpha_1, \ldots, \alpha_{r+1}) \in k^{r+1}$ with $\alpha_j \neq 0$ for some $j \in \{1, \ldots, r\}$.*

(i) $(u_\alpha - 1)^{p-1}\mu_j^{p-1} = c_j v_j^{p-1}(e_j - 1)^{p-1}\mu_j^{p-1}$ *for some non-zero $c_j \in k$ and a unit $v_j = 1 + x_j \in k[G]$.*

(ii) $\nu_{A_{j'}} = (u_\alpha - 1)^{p-1}\mu_j^{p-1} = u_j(e_j - 1)^{p-1}\mu_j^{p-1} = u_j\nu_E$ *for some unit $u_j \in k[G]$ for the group $A_{j'} = \langle e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_r, u_\alpha\rangle$.*

(iii) $\nu_A = 0$.

Proof. Define $(g_i - 1)^{p^{n_i - t}} = g_i - 1$ if $n_i \leq t$. Since $z \in J_G = k[G](g_1 - 1) + \ldots + k[G](g_r - 1)$ and $z^{p^t} = 0$,

$$z \in k[G](g_1 - 1)^{p^{(n_1 - t)}} + \cdots + k[G](g_r - 1)^{p^{(n_r - t)}}.$$

Thus we can write $z = \sum_{i=1}^r \rho_i(g_i - 1)^{p^{(n_i - t)}}$ for some $\rho_1, \ldots, \rho_r$ in J_G, not all zero, as $z \in \mathbb{J}^{(2)}$. Then

$$w = \sum_{i=1}^{r} \rho_i^{p^{(t-1)}}(e_i - 1),$$

not all $\rho_i^{p^{t-1}}$ are zero. Moreover, whenever $i \neq j$, we have $(e_i - 1)\mu_j^{p-1} = 0$ for $i = 1, \ldots, r$. Thus

$$w\mu_j^{p-1} = \rho_j^{p^{(t-1)}}(e_j - 1)\mu_j^{p-1}, \tag{1}$$

and hence

$$\begin{aligned}(u_\alpha - 1)\mu_j^{p-1} &= \Big(\sum_{i=1}^{r} \alpha_i(e_i - 1) + \alpha_{r+1}w\Big)\mu_j^{p-1} \\ &= \mu_j^{p-1}\alpha_j(e_j - 1) + \alpha_{r+1}\rho_j^{p^{(t-1)}}(e_j - 1)\mu_j^{p-1} \\ &= (\alpha_j + \alpha_{r+1}\rho_j^{p^{(t-1)}})(e_j - 1)\mu_j^{p-1}.\end{aligned}$$

Observe that

$$\begin{aligned}(u_\alpha - 1)^2 \mu_j^{p-1} &= \big[(\alpha_j + \alpha_{r+1}\rho_j^{p^{(t-1)}})(e_j-1)\big](u_\alpha - 1)\mu_j^{p-1}\\ &= \big[(\alpha_j + \alpha_{r+1}\rho_j^{p^{(t-1)}})(e_j-1)\big]^2 \mu_j^{p-1},\end{aligned}$$

and finally,

$$\begin{aligned}(u_\alpha - 1)^{p-1} \mu_j^{p-1} &= [\mu_j^{p-1}\alpha_j(e_j-1) + \alpha_{r+1}\rho_j^{p^{(t-1)}}(e_j-1)](u_\alpha - 1)\mu_j^{p-1}\\ &= \big[(\alpha_j + \alpha_{r+1}\rho_j^{p^{(t-1)}})(e_j-1)\big]^{p-1} \mu_j^{p-1}.\end{aligned}$$

Set $v'_j = (\alpha_j + \alpha_{r+1}\rho_j^{p^{(t-1)}})$. Since $\alpha_j \neq 0$, v'_j is a unit in $k[G]$.

Letting $v_j = \alpha_j^{-1} v'_j$ proves part (i) of the lemma. Part (ii) follows from the definitions.

For (iii): Using equation (1), we obtain

$$w^2\mu_j^{p-1} = (\rho_j^{p^{(t-1)}}(e_j-1))w\mu_j^{p-1} = (\rho_j^{p^{(t-1)}}(e_j-1))^2\mu_j^{p-1},$$

and finally,

$$w^{p-1}\mu_j^{p-1} = (\rho_j^{p^{(t-1)}}(e_j-1))^{p-1}\mu_j^{p-1}.$$

Therefore

$$\nu_A = (e_j-1)^{p-1}w^{p-1}\mu_j^{p-1} = (e_j-1)^{2(p-1)}\big(\rho_j^{p^{(t-1)}}\big)^{p-1}\mu_j^{p-1} = 0$$

as $2p-2 \geq p$ for any prime number p and $(e_j-1)^p = 0$. □

Proposition 2.11. *Let x, v, z be pairwise commuting nilpotent operators on $M = k^d$ for k a field of characteristic p. Suppose that the nilpotencies of x, v, z are p, p, l, respectively, $l \geq 1$ Then M is free as a $k[C_p]$-module where the action of $g-1$ on M is given by x if and only if M is free as a $k[C_p]$-module where the action of $g-1$ on M is given by $x+vz$ for a generator g of C_p.*

Proof. Follows from Proposition 2.2 of [**FP**] which is a generalization of Lemma 6.4 of [**Ca**]. In the statement of Proposition 2.2 of [**FP**] z is assumed to have nilpotency p^r instead of l. We observed that the proof uses only that z is nilpotent. □

3. Main Theorems

Let G be an abelian p-group and E be its unique maximal elementary abelian p-subgroup with generators given in Notation 2.4. In order to show that a module is free we will use Lemma 2.1 or Dade's Lemma, or sometimes the equivalent of Dade's Lemma, namely, $V_E^{\mathbf{r}}(M) = 0$.

When G is an elementary abelian p-group Proposition 3.1 is Lemma 6.1 in [**Ca**]. The proof we give here is based on the proof of Lemma 6.1 in [**Ca**] and Lemma 2.10. Rougly speaking, the role of $\nu_{H'}$ in [**Ca**] is played by $u \cdot \nu_{H'}$ where u is a unit of $k[G]$.

Proposition 3.1. *Suppose G is an abelian p-group and $z \in \mathbb{J}^{(2)} \backslash \{0\}$. Then $k[G]$ is not free as a $k[\langle 1+z \rangle]$-module.*

Proof. Let $z \in \mathbb{J}^{(2)}$ with $z \neq 0$. Since $\mathbb{J}^{(2)}$ is contained in J_G, z is nilpotent. By Lemma 2.3, (i) and (ii) $\langle 1+z \rangle \cong C_{p^t}$ for some positive integer t, and $z^{p^{(t-1)}} \neq 0$. By Chouinard's theorem it suffices to show that $k[G]$ is not free as a $k[\langle 1+z^{p^{t-1}} \rangle]$-module. To simplify the notation we set set $w = z^{p^{(t-1)}}$.

Since $w^p = 0$, by Lemma 2.3 (iv) we have that $k[\langle 1+w \rangle] \cong k[C_p]$ if and only if $w^{p-1} \neq 0$. If $w^{p-1} = 0$, then $\dim_k(k[\langle 1+w \rangle]) < p$, however $\dim_k k[G]$ is p^l for some l. Thus $k[G]$ cannot be free as a $k[\langle 1+w \rangle]$-module. Hence we can assume that $w^{p-1} \neq 0$, thus $k[\langle 1+w \rangle] \cong k[C_p]$. (Note that it is still possible that $k[\langle 1+z \rangle] \ncong k[C_{p^t}]$, see 2.7 (1).) Let $H = \langle f_1, \ldots, f_{r+1} \rangle$ be an elementary abelian p-group of order p^{r+1}. Define an action of H on $k[G]$ as follows; $f_i m = e_i m$ for all $i = 1, \ldots, r$, and $f_{r+1} m = (1+w)m$ for all $m \in k[G]$.

Claim: $V_H^{\mathbf{r}}(k[G]) = k\{(0, \ldots, 0, 1)\} \subseteq k^{r+1}$.

Since $\nu_{\langle f_1, \ldots, f_r \rangle} k[G] = \nu_E k[G]$, and $k[G]{\downarrow_E}$ is free as a $k[E]$-module (E is a subgroup of G) we have $k[G]{\downarrow_{\langle f_1, \ldots, f_r \rangle}}$ is free. Hence by Dade's Lemma none of $\alpha = (\alpha_1, \ldots, \alpha_r, 0) \in k^{r+1} \backslash 0$ is in $V_H^{\mathbf{r}}(k[G])$.

Let $\alpha = (\alpha_1, \ldots, \alpha_{r+1}) \in k^{r+1}$ with $\alpha_j \neq 0$ for some $j = 1, \ldots, r$,

$$u_\alpha = 1 + \sum_{i=1}^{r+1} \alpha_i (f_i - 1) \quad \text{and} \quad u'_\alpha = 1 + \sum_{i=1}^{r} \alpha_i (e_i - 1) + \alpha_{r+1} w.$$

Then $\langle u_\alpha \rangle$ and $\langle u'_\alpha \rangle$ are shifted cyclic subgroups of $k[H]$ and $k[A]$ respectively where $A = \langle e_1, \ldots, e_r, 1+w \rangle$. To show $k[G]{\downarrow_{\langle u_\alpha \rangle}}$ is free it suffices to show that $k[G]{\downarrow_{H_{j'}}}$ is free where $H_{j'} = \langle f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_r, u_\alpha \rangle$. By Lemma 2.10(ii) $\nu_{A_{j'}} = u_j \nu_E$ for some unit u_j and the subgroup $A_{j'} = \langle e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_r, u'_\alpha \rangle$ of $k[G]$. Since $\nu_{H_{j'}} k[G] = \nu_{A_{j'}} k[G]$ and $k[G]{\downarrow_E}$ is free, we obtain that $k[G]{\downarrow_{H_{j'}}}$ is free. Thus $k[G]{\downarrow_{\langle u_\alpha \rangle}}$ is free. Therefore $\alpha \notin V_H^{\mathbf{r}}(k[G])$ for $\alpha = (\alpha_1, \ldots, \alpha_{r+1})$ with $\alpha_j \neq 0$ for some $j = 1, \ldots, r$. Thus $V_H^{\mathbf{r}}(k[G]) \subseteq k\{(0, \ldots, 0, 1)\} \subseteq k^{r+1}$.

On the other hand $\nu_A = 0$ by Lemma 2.10(iii). Hence $\nu_H k[G] = \nu_A k[G] = 0$. Therefore $k[G]$ is not free as a $k[H]$-module. Thus $V_H^{\mathbf{r}}(k[G]) \neq 0$. Then $V_H^{\mathbf{r}}(k[G]) =$

$k\{(0,\ldots,0,1)\}$. This proves the Claim. The statement that $k[G]$ is not free as a $k[\langle 1+w\rangle]$-module follows from the Claim. □

It can happen that even if $x \notin \mathbb{J}^{(2)}$, $k[G]\downarrow_{\langle 1+x\rangle}$ is not free. That is, the converse of Proposition 3.1 is not true. Let G and x be as in Example 2.7 (4). We have $x \notin \mathbb{J}^{(2)}$, but $x^3 \in \mathbb{J}^{(2)}$. Thus $k[G]\downarrow_{\langle 1+x^3\rangle}$ is not free by the above propostion (or as $\dim(x^6k[G]) = 9 < 27 = |G|/3)$, hence $k[G]\downarrow_{\langle 1+x\rangle}$ is not free by Chouinard's Theorem. Observe that, $x_L = g-1$ and $|\langle 1+x_L\rangle| = 3$ whereas $|\langle 1+x\rangle| = 9$. The next theorem gives a necessary and sufficient condition for the equality $|\langle 1+x_L\rangle| = |\langle 1+x\rangle|$.

Theorem 3.2. *Suppose that G is an abelian p-group and $x \in J_G$. Then $k[G]$ is free as a $k[\langle 1+x\rangle]$-module if and only if $x^{|\langle 1+x\rangle|/p} \notin \mathbb{J}^{(2)}$ if and only if $|\langle 1+x_L\rangle| = |\langle 1+x\rangle|$.*

Proof. Suppose $x \in J$. By Lemma 2.3 (ii) we know that $\langle 1+x\rangle \cong C_{p^t}$ for some $t \geq 0$, and $x^{p^t} = 0$, $x^{p^{(t-1)}} \neq 0$. Define $(g_i-1)^{p^{(n_i-t)}} = (g_i-1)$ when $n_i - t \leq 0$, and $y = x^{p^{t-1}}$. As in Lemma 2.10, since

$$x \in k[G](g_1-1)^{p^{(n_1-t)}} + \cdots + k[G](g_r-1)^{p^{(n_r-t)}},$$

we can write

$$x = s_1(g_1-1)^{p^{(n_1-t)}} + \cdots + s_r(g_r-1)^{p^{(n_r-t)}}$$

for some $s_1,\ldots,s_r$ in $k[G]$. Taking $p^{(t-1)}$-th power gives

$$y = x^{p^{(t-1)}} = s_1^{p^{(t-1)}}(g_1-1)^{p^{(n_1-1)}} + \cdots + s_r^{p^{(t-1)}}(g_r-1)^{p^{(n_r-1)}}.$$

To simplify the notation write $s_i' = s_i^{p^{(t-1)}}$ for $i = 1,\ldots,r$. Then

$$y = s_1'(e_1-1) + \cdots + s_r'(e_r-1).$$

$\Longleftarrow$: Assume that $y \notin \mathbb{J}^{(2)}$. The assumption $y \notin \mathbb{J}^{(2)}$ implies that there exists a j in $\{1,\ldots,r\}$ such that $s_j = a_j + w_j$ for some non-zero $a_j \in k$, and $w_j \in J$. Thus $s_j' = a_j' + w_j'$ for a non-zero $a_j' = a_j^{p^{(t-1)}} \in k$, and $w_j' = w_j^{p^{(t-1)}} \in J$. Define an elementary abelian subgroup of units of $k[G]$ as $K_j = \langle e_1,\ldots,e_{j-1},e_{j+1},\ldots,e_r,1+y\rangle$. Note that $K_j \cong E$ and $\nu_E = \mu^{p-1}$ where μ is as defined in Notation 2.9.

Claim: $\nu_{K_j} = a_j^{p^t - p^{(t-1)}} v_j \nu_E$ for some unit v_j in $k[G]$. We argue as in the proof of Lemma 2.10. Note that $(e_i-1)\mu_j^{p-1} = 0$ for $i = 1,\ldots,r$ except for $i \neq j$. Then

$$y\mu_j^{p-1} = s_j'(e_j-1)\mu_j^{p-1}.$$

Hence $y^2\mu_j^{p-1} = (s_j'(e_j-1))^2\mu_j^{p-1}$ and eventually

$$\nu_{K_j} = y^{p-1}\mu_j^{p-1} = (s_j'(e_j-1))^{p-1}\mu_j^{p-1} = (s_j')^{p-1}\mu^{p-1} = (s_j')^{p-1}\nu_E.$$

Since $s_j' = a_j' + w_j'$, Lemma 2.2 implies that

$$(s_j')^{p-1} = (a_j')^{p-1} + (p-1)(a_j')^{p-2}w_j' + (a_j')^{p-3}(w_j')^2 + (p-1)(a_j')^{p-4}(w_j')^3 + \cdots + (p-1)a_j'(w_j')^{p-2} + (w_j')^{p-1}.$$

Therefore $(s_j')^{p-1} = (a_j')^{p-1}(1+z_j)$ for some $z_j \in J$. Letting $v_j = 1+z_j$ proves the Claim.

Using the Claim we obtain that $\nu_{K_j}k[G] = \nu_E k[G]$, and hence

$$\dim(\nu_{K_j}k[G]) = \dim(\nu_E k[G]).$$

Therefore $k[G]$ is a free $k[K_j]$-module by Lemma 2.1. Since $\langle 1+y\rangle$ is a subgroup of K_j, $k[G]$ is free as a $k[\langle 1+y\rangle]$-module.

$\Longrightarrow$: Assume that $k[G]$ is free as $k[\langle 1+x\rangle]$-module. Then $k[G]$ is free as $k[\langle 1+y\rangle]$-module by Chouinard's Theorem. Therefore $y \notin \mathbb{J}^{(2)}$ by Proposition 3.1. The last assertion of the theorem is true by Lemma 2.8 (ii). $\square$

Corollary 3.3. *Let $x \in J$. Then x is a p^t-point if and only if $x^{p^{t-1}}$ is a p-point.*

In the special case of $G = C_2 \times C_4$, for x in J, x is in $\mathbb{J}^{(2)}$ if and only if $x^{|\langle 1+x\rangle|/2}$ is in $\mathbb{J}^{(2)}$ [**Ka**]. However this is no longer the case for groups of higher order, such as $C_2 \times C_8$ or $C_3 \times C_{27}$, see 2.7 (2), (4).

Although we work with finite abelian p-groups we make some of our definitions in the more general setting of finite group schemes. For a finite group scheme $\mathcal{G}$, following the notation of [**FP**], $k[\mathcal{G}]$ denotes the coordinate ring of $\mathcal{G}$, $k\mathcal{G}$ denotes the k-linear dual algebra of $k[\mathcal{G}]$.

Definition 3.4. Let $\mathcal{G}$ be a finite group scheme over k. A flat map $\phi : k[X]/(X^{p^t}) \longrightarrow k\mathcal{G}$ of algebras is called a *p^t-point* or a *p-power point of degree t* of $k\mathcal{G}$ if ϕ factors through $k\mathcal{C}$ for some abelian unipotent subgroup scheme $\mathcal{C}$ of $\mathcal{G}$. A p-power point (or a p^*-point) is a p-power point of degree t for some t.

When $t=1$ the above definition is given in [**FP**]. For a finite abelian p-group G, x in J_G, a positive integer t, by Theorem 3.2 we have that x is a *p^t-point* or a *p-power point of degree t* of $k[G]$ provided that $\langle 1+x\rangle \cong C_{p^t}$ and $x^{p^{(t-1)}}$ is not in $\mathbb{J}^{(2)}$. A *p-power point* or a *p^*-point*, is a p^t-point for some t. That is, x is a p-power point if and only if $\langle 1+x\rangle$ is a shifted cyclic subgroup. *The set of all* *p-power points of degree t in $k[G]$ is denoted by $P^t(G)$.* The set of all p^t-points of $k[G]$ for all $t = 1, \dots, m$ is denoted by $P^*(G)$. Denote by $\mathcal{S}^t(G)$ and $\mathcal{S}^*(G)$ the sets of all shifted cyclic subgroups of order p^t and of all shifted cyclic subgroups of $k[G]$, respectively. For N^t given in 2.4, ${\mathbb{J}^{(2)}}^t$ given in 2.5 we have

$$P^*(G) = \{x \in J \mid x^{|\langle 1+x\rangle|/p} \notin \mathbb{J}^{(2)}\}, \qquad S^*(G) = \{\langle 1+x\rangle \mid x \in P^*(G)\},$$

$$P^t(G) = \{x \in J \mid x^{|\langle 1+x\rangle|/p} \notin \mathbb{J}^{(2)} \text{ and } \langle 1+x\rangle \cong C_{p^t}\}, \quad \text{or} \quad P^t(G) = N^t \oplus {\mathbb{J}^{(2)}}^t$$

and

$$S^t(G) = \{\langle 1+x\rangle \mid x \in P^t(G)\}.$$

The following theorem is a generalization of Lemma 6.4 in [**Ca**] and also of Lemma 1.5 (1) in [**Fa**], as well as Theorem 4.3 in [**Ka**].

Theorem 3.5. *Let x, y be in $P^t(G)$ and M be a finitely generated $k[G]$-module. If $x \equiv y \mod (\mathbb{J}^{(2)})$, then $M{\downarrow}_{\langle 1+x\rangle}$ is free if and only if $M{\downarrow}_{\langle 1+y\rangle}$ is free.*

Proof. By the hypothesis $x \equiv y \mod (\mathbb{J}^{(2)})$ we can write $y = x + w$ where $w \in \mathbb{J}^{(2)}$. Since x, y are in $P^t(G)$, the nilpotencies of x and y are p^t, $w^{p^t} = 0$, $x^{p^{(t-1)}} \notin \mathbb{J}^{(2)}$, $y^{p^{(t-1)}} \notin \mathbb{J}^{(2)}$. By Chouinard's Theorem it suffices to show that $M{\downarrow}_{\langle 1+x^{p^{(t-1)}}\rangle}$ is free if and only if $M{\downarrow}_{\langle 1+y^{p^{(t-1)}}\rangle}$ is free. By Lemma 2.8 (i) we know $x^{p^{(t-1)}} \equiv y^{p^{(t-1)}} \mod (\mathbb{J}^{(2)})$. Also, the nilpotencies of $x^{p^{(t-1)}}$ and $y^{p^{(t-1)}}$ are p, and $w^{p^{(t-1)}}$ is nilpotent. Since $w^{p^t} = 0$, we have $w \in (g_1-1)^{p^{(n_1-t)}} J_G + \dots + (g_r-1)^{p^{(n_r-t)}} J_G$. We can write $w = x_1y_1 + \dots + x_ry_r$ for some $x_1, \dots, x_r, y_1, \dots, y_r$ in J_G not necessarily distinct, $x_i \in (g_i-1)^{p^{(n_i-t)}}$. Since $(w^{p^{(t-1)}})^p = 0$, we can write $w^{p^{(t-1)}} = v_1z_1 + \dots + v_sz_s$ where v_i is p-nilpotent, and z_i is nilpotent for $i = 1, \dots, s$, for some $s \le r$. The statement then follows by the repeated application of Proposition 2.11. Namely, first apply it to the triple $x^{p^{(t-1)}}$, v_1, z_1, then to the triple $x^{p^{(t-1)}} + v_1z_1$, v_2, z_2, and finally to the triple $x^{p^{(t-1)}} + v_1z_1 + \dots + v_{s-1}z_{s-1}$, v_s, z_s, to obtain that $M{\downarrow}_{\langle 1+x^{p^{(t-1)}}\rangle}$ is free if and only if $M{\downarrow}_{\langle 1+y^{p^{(t-1)}}\rangle}$ is free. □

Definition 3.6. Define an equivalence relation on $P^t(G)$ by setting $x \sim y$ if and only if $M{\downarrow}_{\langle 1+x\rangle}$ is free if and only if $M{\downarrow}_{\langle 1+y\rangle}$ is free for every finitely generated $k[G]$-module M. Let $\mathcal{P}^t(G)$ denote the quotient set of this relation.

By Theorem 3.5 the map $P^t(G)/{\mathbb{J}^{(2)}}^t \longrightarrow \mathcal{P}^t(G)$ taking $\bar{x}$ to $[x]$ is well-defined and surjective but not necessarily injective as shown by the example in the remark below. Thus the converse of Theorem 3.5 is not necessarily true.

Remark 3.7. Note that there is a one to one correspondence with $\mathcal{P}^1(G)$ and $P(G)$ of [**FP**]. When the group is elementary abelian we can state it as follows; the lines through the origin in J/J^2 correspond to the equivalence classes of shifted cyclic subgroups [**Ca**]. However when the group G is non-elementary abelian the map from $J/\mathbb{J}^{(2)}$ to $\mathcal{P}^t(G)$ given by $\bar{x} \longrightarrow [x_{L^1} + \cdots + x_{L^t}]$ is only surjective not necessarily injective. For an example having non-trivial kernel consider the smallest non-elementary abelian 2-group $G = C_2 \times C_4$ with generators g, h of degrees 2, 4 respectively. We have

$$\mathcal{P}^1(G) = \{[a(g-1) + b(h-1)^2 + w] \mid (a, b) \neq (0,0), a,\, b \in k,\ w \in \mathbb{J}^{(2)}\},$$

$$\mathcal{P}^2(G) = \{[a(g-1) + b(h-1)^2 + c(h-1) + w] \mid c \neq 0, a,\, b,\, c \in k,\ w \in \mathbb{J}^{(2)}\}.$$

The distinct, non-collinear points $\alpha = (0,1,1)$, and $\beta = (0,0,1)$ of $k^3 \cong J/\mathbb{J}^{(2)} \cong k(e-1) + k(f-1)^2 + k(f-1)$ give rise to the same group algebra $k[\langle 1 + x_\alpha \rangle] = k[\langle 1 + x_\beta \rangle]$ for $x_\alpha = f - 1 + (f-1)^2$, $x_\beta = f - 1$ in $P^2(G)$. Therefore the correspondence between the lines through the origin in $k^3 \cong J/\mathbb{J}^{(2)}$ and the shifted cyclic subgroups of $k[G]$ is not one-to-one for a non-elementary abelian p-group. This example also shows that $x \sim y$ does not imply $x \equiv y \mod (\mathbb{J}^{(2)})$.

When the group G is elementary abelian Proposition 3.8, 3.9 are proved in [**Ca**, Lemma 2.23, Lemma 6.3]. Although they are not explicitly referred in this article we included them to show that shifted subgroups behave as ordinary subgroups of G in some respect.

Proposition 3.8. *Let G be an abelian p-group, H be a p-subgroup of the group* Units($k[G]$)*, and M, S be finitely generated $k[G]$-modules. If the restriction $M{\downarrow_H}$ is free, then the restriction $(M \otimes S){\downarrow_H}$ is free.*

Proof. Let H be a p-subgroup of the group Units($k[G]$). Note that the assertion is trivially true if H is a subgroup of G. We proceed by induction on the dimension of S, set $d = \dim(S)$. If d is equal to 1, then $M \otimes S \cong M$ hence there is nothing to show. Suppose d is bigger than 1. Since G is a p-group, there exists a submodule N in S with the property that N is isomorphic to the trivial module k. Then the restrictions $(M \otimes N){\downarrow_H}$ and $(M \otimes (S/N)){\downarrow_H}$ are free by the induction hypothesis.

Thus the short exact sequence

$$0 \longrightarrow (M \otimes N){\downarrow_H} \longrightarrow (M \otimes S){\downarrow_H} \longrightarrow (M \otimes (S/N)){\downarrow_H} \longrightarrow 0$$

is split exact. Therefore the restriction $(M \otimes S){\downarrow_H}$ is free. $\square$

Proposition 3.9. *Let G be an abelian p-group and H be a p-subgroup of the group* $\mathrm{Units}(k[G])$. *The following are equivalent.*

(i) *The restricted module $k[G]{\downarrow_H}$ is free.*

(ii) *There exists a $k[G]$-module M such that the restricted module $M{\downarrow_H}$ is free.*

Proof. It is obvious that (i) implies (ii). For the converse; assume that M is a $k[G]$-module which is free as a $k[H]$-module. Then by Proposition 3.8 we know that the restriction $(M \otimes k[G]){\downarrow_H}$ is free. On the other hand, the module $M \otimes k[G]$ is necessarily free as a $k[G]$-module. Hence $(M \otimes k[G]){\downarrow_H} \cong (\oplus^r k[G]){\downarrow_H}$ for some r as $k[H]$-modules, and consequently $(\oplus^r k[G]){\downarrow_H} \cong \oplus^s k[H]$ for some s as $k[H]$-modules. The conclusion that $k[G]{\downarrow_H}$ is free follows from Krull-Schmidt Theorem since $k[H]$ is an indecomposable $k[H]$-module due to the fact that H is a p-group. □

4. Modules of constant p-power Jordan type

Let G be an abelian p-group and E be its unique maximal elementary abelian p-subgroup with generators given in Notation 2.4 as usual. The immediate examples of $k[G]$-modules of constant p-power Jordan type are k and $k[G]$. The modules L_{ζ_α}, known as Carlson modules, are not of constant Jordan type hence not of constant p-power Jordan type, for their definition see [**Be2**]. Because they are free at every point except α [**Ca**]. When G is elementary abelian modules of constant Jordan type and modules of constant p-power Jordan type coincide as there are only cyclic subgroups of order p. Let $\langle g \rangle \cong C_{p^t}$, and define $J^0_{\langle g \rangle} = k[\langle g \rangle]$ and $J_{\langle g \rangle} = Rad(k[\langle g \rangle])$. It is well-known that $J^i_{\langle g \rangle}$ is an indecomposable $k[\langle g \rangle]$-module of $\dim_k(J^i_{\langle g \rangle}) = p^t - i$ for $i = 0, 1, \dots, p^t - 1$, and any indecomposable $k[\langle g \rangle]$-module is isomorphic to one of them. Note that $J^{p^t-1}_{\langle g \rangle} \cong k\nu_{\langle g \rangle}$ is the trivial module. Hence, for a p^t-point x of $k[G]$ and a $k[G]$-module M, we have

$$M{\downarrow_{\langle 1+x \rangle}} \cong \sum_{l=0}^{p^t-1} (J^l_{\langle 1+x \rangle})^{\eta_{p^t-l}(x)},$$

where $\eta_i(x)$ is the number of i-dimensional indecomposable summands of $M{\downarrow_{\langle 1+x \rangle}}$. Thus the decomposition of $M{\downarrow_{\langle 1+x \rangle}}$ can be represented by

$$\underline{\eta}^t = \eta_{p^t}[p^t] + \cdots + \eta_1[1].$$

That is, a $k[\langle 1+x \rangle]$-module M is completely determined by the Jordan canonical form of the matrix representing the action of x on M which we refer as the *Jordan type* of M at x as in [**CFP**]. When x is a p^t-point, $\eta_{p^t}[p^t] + \cdots + \eta_1[1]$ denotes the

Jordan type of x where $[i]$ denotes the Jordan block of size $i \times i$, η_i denotes the number of $[i]$. The formula given below is used to compute η_i provided that the matrix X representing the action of x is known

$$\eta_i = \operatorname{rank}(X^{i-1}) - 2\operatorname{rank}(X^i) + \operatorname{rank}(X^{i+1}).$$

Definition 4.1. The *Jordan type of M at a p^t-point x*, or *the p^t-Jordan type of M at a p^t-point x*, denoted by p^t-Jtype$(M\downarrow_{\langle 1+x\rangle})$, is defined as $\underline{\eta}^t$. We say that M is *of constant p^t-Jordan type*, and refer to $\underline{\eta}^t$ as *the p^t-Jordan type of M*, provided that p^t-Jtype$(M\downarrow_{\langle 1+x\rangle})$ is the same for every x in $P^t(G)$. In that case we simply write p^t-Jtype(M). We say M is of *constant p^t-power Jordan type*, if M is of constant p^l-Jordan type for all $l = 1, \dots, t$. A module of constant p^m-power Jordan type is referred as a module of *constant p-power-Jordan type* for simplicity.

The above definition could be made for any $\mathcal{G}$-module M for a finite group scheme $\mathcal{G}$ over k as well by requiring p^t-Jordan type of M to be the same at every p^t-point of $\mathcal{G}$ as defined in 3.4.

Remark 4.2. There is no reason for the i-times repeated Frobenius map $F^i : \mathcal{P}^t(G) \longrightarrow \mathcal{P}^{t-i}(G)$ to be surjective or injective for any $i = 1, \dots, m$. In particular, F from $\mathcal{P}^t(G)$ to $\mathcal{P}^{t-1}(G)$ is not surjective. Thus, although the p^t-Jordan type of x determines the p^{t-1}-Jordan type of x^p, we cannot say that a module of constant p^t-Jordan type is necessarily a module of constant p^{t-1}-Jordan type, see 4.3 for an example.

Example 4.3. Let $C_3 \times C_9 = \langle g, h\rangle$ where g, h are of orders 3, 9 respectively, and let $\swarrow$ denote the action of $g-1$ and $\searrow$ denote the action of $h-1$ on M. Let M be the $k[C_3 \times C_9]$-module given by Figure 1. It can be computed that
9-JType$(M\downarrow_{\langle h\rangle}) = [3] + [2] + [1]$,
3-JType$(M\downarrow_{\langle h^3\rangle}) = 6[1]$,
3-JType$(M\downarrow_{\langle g\rangle}) = [3] + [2] + [1]$.
Obviously M is not of constant 3-type, whereas it can be shown that M is of constant 9-Jordan type. That is M is in C_G^2 but not in C_G^1 (so that it is not in $\mathcal{C}_G^2$).

4.1. **A Filtration for Modules of Constant Jordan Type.** There is a decreasing filtration of modules of constant Jordan type having the set of modules of constant p-power Jordan type as the last term;

$$\mathcal{C}_G^1 \supseteq \mathcal{C}_G^2 \supseteq \cdots \supseteq \mathcal{C}_G^m \tag{2}$$

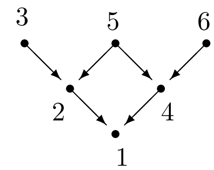

Figure 1.

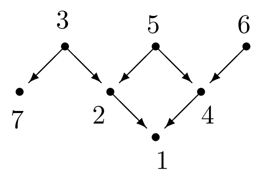

Figure 2.

with $\mathcal{C}_G^t$ denoting the set of all $k[G]$-modules of constant p^l-Jordan type for $l = 1, 2, \dots, t$. For an example of a module in $\mathcal{C}_G^t$ but not in $\mathcal{C}_G^{t+1}$ see Example 4.4. Note that a $k[G]$-module M is of constant p^t-power Jordan type if and only if $\Gamma^i(G)_M = \emptyset$ for $i = 1, \dots, t$, see Introduction for the definition of $\Gamma^i(G)_M$.

Example 4.4. There is a $k[C_4 \times C_4]$-module M which is of constant 2-Jordan type, but not of constant 4-Jordan type, i.e., $M \in \mathcal{C}_G^1 \backslash \mathcal{C}_G^2$. Let $C_4 \times C_4 = \langle g, h \rangle$ where g, h are of order 4. Let $\swarrow$ denote the action of $g - 1$ and $\searrow$ denote the action of $h - 1$ on M, and M be given by the diagram in Figure 2. It can be computed that
2-JType$(M) = [2] + 5[1]$, and
4-JType$(M{\downarrow_{\langle h \rangle}}) = [3] + [2] + 2[1]$, whereas
4-JType$(M{\downarrow_{\langle g \rangle}}) = [3] + 2[2]$.

It is desirable to obtain a method of constructing examples of modules which are in $\mathcal{C}_G^i \setminus \mathcal{C}_G^{i+1}$ firstly for a given i, for the group $C_{p^m} \times C_{p^m}$, eventually for any finite abelian p-group.

4.2. **Endotrivial Modules.** As shown by Dade in [**Da**] the indecomposable endotrivial modules for $k[G]$ are of the form $\Omega_G^n(k)$ for $n \in \mathbb{Z}$ where $\Omega_G^n(k)$ is the n-th Heller shift of the trivial $k[G]$-module k. These modules have the property that $\Omega_G^n(k){\downarrow_{\langle 1+x \rangle}} \cong \Omega_{\langle 1+x \rangle}^n(k) \oplus \text{free}$ for any p-power point x. Hence their Jordan type is

$$l[p^t] + 1[1] \qquad \text{or} \qquad l[p^t] + 1[p^t - 1]$$

for some l respectively for even n and odd n. This proves the following theorem.

Theorem 4.5. *Endotrivial $k[G]$-modules are are of constant p-power Jordan type.*

4.3. **Modules with Equal Image Property.** Inspired by the definition of modules of constant image property given in [**CF**] for elementary abelian p-groups we adopt the definition to our setting as follows. These modules are renamed as modules of equal image property in the subsequent work [**CFS**].

Definition 4.6. A $k[G]$-module M is said to have the equal images property if $xM = yM$ for all x, y in $P^t(G)$ for all $t = 1, \dots, m$.

The following theorem follows from the definitions.

Theorem 4.7. *$k[G]$-modules of equal images property are modules of constant p-power Jordan type.*

4.4. **Examples of Constant p-power Jordan Type Modules.** Recall that a $k[G]$-module M is of constant p^t-power Jordan type if and only if $\Gamma^i(G)_M = \emptyset$ for $i = 1, \dots, t$.

Definition 4.8. Two constant p^t-power Jordan type modules M and M' are called *p^t-power Jordan type equivalent* provided that $M{\downarrow_{\langle 1+x\rangle}}$ has the same decomposition as $M'{\downarrow_{\langle 1+x\rangle}}$ for all x in $P^i(G)$, for $i = 1, \dots, t$. When M and M' are p^m-power Jordan type equivalent we simply call them as *Jordan type equivalent*. Let $\mathcal{J}_G^t$ denote the quotient set of the equivalence relation $\sim$ defined on $\mathcal{C}_G^t$ by setting $M \sim M'$ if and only if M and M' are p^t-power Jordan type equivalent.

In Examples 4.9-4.11 we present a pair of non-isomorphic $k[G]$-modules M , M' of constant p-power Jordan type such that $[M]$ and $[M']$ are the same as elements of $\mathcal{J}_G^i$ but $[M]$ and $[M']$ are not the same as elements of $\mathcal{J}_G^{i+1}$, for $G = C_2 \times C_4$, $C_8 \times C_8$ and $C_4 \times C_4$. In each case let the group $G = \langle g, h\rangle$ and let $\swarrow$ denote the action of $g-1$, and $\searrow$ denote the action of $h-1$ in the figures.

Example 4.9. There are non isomorphic $k[C_2 \times C_4]$-modules M and M', given in Figure 3 and Figure 4 respectively. Each one of them is of constant 2-power Jordan type, i.e., M has constant 2-Jtype and has constant 4-Jtype, similarly for M'. But M and M' have different Jordan types at 4-points even though they have the same Jordan type at 2-points. That is, $[M] = [M']$ in $\mathcal{J}_G^1$ but $[M] \neq [M']$ in $\mathcal{J}_G^2$. Hence M and M' are necessarily of constant Jordan type with the same Jordan type (at 2-points) so that M and M' are not distinguishable if, only 2-points are used. It

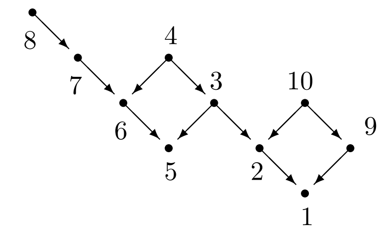

Figure 3.

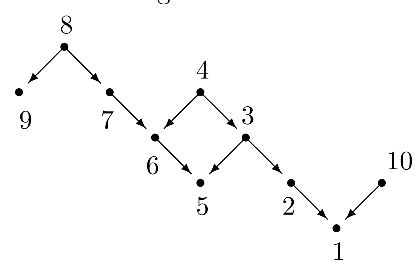

Figure 4.

can be computed that
4-JType$(M) = 2[4] + 1[2]$ and
4-JType$(M') = 2[4] + 2[1]$,
2-JType$(M) =$ 2-Jype$(M') = 4[2] + 2[1]$.

Example 4.10. There are non isomorphic $k[C_8 \times C_8]$-modules M and M', given in Figure 5, Figure 6 respectively, which are of constant 2-power Jordan type, but have different Jordan types at 8-points, even though they have the same Jordan type at 2-points and 4-points. That is, $[M] = [M']$ in $\mathcal{J}_G^1$, $[M] = [M']$ in $\mathcal{J}_G^2$, but $[M] \neq [M']$ in $\mathcal{J}_G^3$. It can be computed that
2-JType$(M) =$ 2-JType$(M') = 12[1]$,
4-JType$(M) =$ 4-JType$(M') = 3[2] + 6[1]$,
8-JType$(M) = 3[3] + [2] + [1]$, but 8-JType$(M') = [4] + [3] + 2[2] + [1]$.

Example 4.11. There are non isomorphic $k[C_4 \times C_4]$-modules M and M' which are of constant 2-power Jordan type, but having different Jordan types at 4-points, and having the same Jordan type at 2-points. That is, $[M] = [M']$ in $\mathcal{J}_G^1$, but $[M] \neq [M']$ in $\mathcal{J}_G^2$. (M and M' are necessarily of constant Jordan type with the same Jordan type.) Note that we can use the modules M and M' of the previous example given in Figure 5 and Figure 6 even though the groups are not the same. It can be computed that

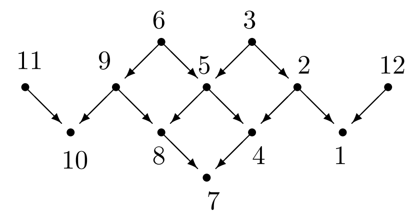

Figure 5.

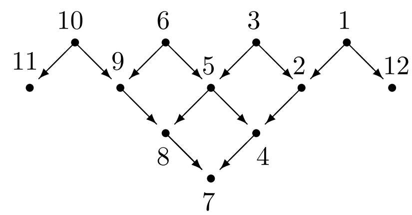

Figure 6.

$$2\text{-JType}(M) = 2\text{-JType}(M') = 3[2] + 6[1],$$
$$4\text{-JType}(M) = 3[3] + [2] + [1], \text{ but } 4\text{-JType}(M') = [4] + [3] + 2[2] + [1].$$

This example shows also that the Jordan type at an 8-point of the previous example is the Jordan type at a 4-point of this example, and similarly, the Jordan type at a 4-point of the previous example is the Jordan type at a 2-point of this example.

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