RESTRICTED MODULES AND CONJECTURES FOR MODULES OF CONSTANT JORDAN TYPE

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Abstract. We introduce the class of restricted $k[A]$-modules and p^t-Jordan types for a finite abelian p-group A of exponent at least p^t and a field k of characteristic p. For these modules, we generalize several theorems by Benson, verify a generalization of conjectures stated by Suslin and Rickard giving constraints on Jordan types for modules of constant Jordan type when t is 1. We state conjectures giving constraints on p^t-Jordan types and show that many p^t-Jordan types are realizable.

1. Introduction

We study finitely generated modules over group algebras of finite abelian p-groups with coefficients in a field k of prime characteristic p. Throughout G, A, E denote abelian p-groups where G, A are of exponent p^t, and E is of exponent p. Studying a $k[A]$-module M via its restrictions to subalgebras $k[C_{p^t}]$ of $k[A]$, for cyclic subgroups C_{p^t} of the unit group of $k[A]$ with the property that $k[A]$ is free as a $k[C_{p^t}]$-module, became a useful tool especially after the work of Carlson [**Ca**] for $t = 1$ followed by [**Ka**] for $t \geq 1$. The decomposition of M as a $k[C_{p^t}]$-module determines the decomposition of the restriction $M{\downarrow}_{C_{p^r}}$ of M, for C_{p^r} contained in C_{p^t}. This fact is the essence of the article.

Up to isomorphism, a $k[C_{p^t}]$-module is completely determined by the multiplicities of the i-dimensional indecomposable summands in its decomposition in terms of indecomposable modules for $i = 1, \dots, p^t$

Definition. Let M be a $k[A]$-module. The p^t-tuple $\underline{b} = (b_1, \dots, b_{p^t})$ with b_i denoting the number of Jordan blocks of size i in the Jordan canonical form of a matrix representing the action of the generator g of a subgroup C_{p^t} of the unit group of $k[A]$ is called *the* p^t*-Jordan type of* M *(at* $g-1$*)* or simply *a Jordan type of* M. The p^{t-s}-Jordan type of $M{\downarrow}_{\langle g^{p^s}\rangle}$ is denoted by $\underline{b}{\downarrow}_{p^s}$ and referred as a p^s*-restricted*

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p^{t-s}*-Jordan type* for $0 < s < t$.

Throughout, unless it is stated otherwise, $\underline{b}$ denotes a p^t-Jordan type of a $k[A]$-module M, and $\underline{a}$ denotes the p^s-restricted p^r-Jordan type $\underline{b}\downarrow_{p^s}$ for some $s > 0$ and $r = t - s$. In this case, $\underline{a}$ is determined by $\underline{b}$, see Proposition 3.1, each a_i is a linear combination of b_j's with coefficients $1, \ldots, p^s$ forming a symmetric pattern as demonstrated in Example 2.2 for $p^t = 5^2$, $s = 1$.

Note that the C_{p^t}'s in the unit group of $k[A]$ are of the form $\langle 1 + x\rangle$ for some x in the Jacobson radical J_A of $k[A]$ with $x^{p^{t-1}} \neq 0$ and $x^{p^t} = 0$.

Definition. An element x in the Jacobson radical J_A of $k[A]$ is called *a* p^r*-point of* A provided that $k[\langle 1 + x\rangle]$ is isomorphic to $k[C_{p^r}]$ and $k[A]\downarrow_{\langle 1+x\rangle}$ is a free module. A p^r-point x of A is called p^s*-restricted (from* G*)* if there is a p^{r+s}-point y (of some group G containing A) with $x = y^{p^s}$.

A sufficient condition for a group A to have a p^s-restricted p^r-point is that, there is a group G containing A as a subgroup of index at most p^s and an element g in G of order p^{r+s} for some $r \geq 1$, so that $g^{p^s} - 1$ is a p^s-restricted p^r-point of A. If A has an element of order p^{r+s}, then $G = A$. Hence, every abelian group A of exponent bigger than p has a p-restricted p-point. As a specific example, let $G = \{1, e, f, ef, f^2, ef^2, f^3, ef^3\}$ be the group $C_2 \times C_4$ and $A = \{1, e, f^2, ef^2\}$ be the maximal elementary abelian subgroup of G. In this setting $f^2 - 1$ is the square of $f - 1$ (also of $f - 1 + (ef^2 - 1)(e - 1)$ etc.), hence it is a 2-restricted 2-point of A.

Definition. A $k[A]$-module M is said to be of *constant* p^t*-Jordan type* provided that the p^t-Jordan type of M at x is the same for every p^t-point x of A. A $k[A]$-module M of constant p^t-Jordan type is called p^s*-restricted* if its p^r-Jordan type is p^s-restricted.

Alternatively, a $k[A]$-module M of constant p^t-Jordan type is called p^s*-restricted (from* $k[G]$*)* if there is an abelian p-group G containing A, such that A has a p^t-point p^s-restricted (from G), and a $k[G]$-module N with $N\downarrow_A$ isomorphic to M. Thus if A has a p^s-restricted p^t-point, then $G = A$. Besides, A having a p^s-restricted p^t-point is equivalent to M having a p^s-restricted p^t-Jordan type. To clearify the notion, suppose M is of constant p^t-Jordan type $\underline{b}$ and also of constant p^r-Jordan type $\underline{a}$ for some $r \leq t$. Then the p^{t-r}-restricted p^r-Jordan type $\underline{b}\downarrow_{p^{t-r}}$ is necessarily equal to $\underline{a}$.

The definition of a p-point and a $k[E]$-module of constant Jordan type are introduced and studied in [**FP**], [**CFS**], [**CFP**]. In fact, the numerous interesting questions and conjectures stated in [**CFP**], especially Suslin's conjecture, for a $k[E]$-module of constant Jordan type motivated this work. We should note here that in a recent article [**CFP1**], $k[E]$-modules of constant radical and socle type are introduced as generalizations of modules of constant Jordan type in a different direction than ours. It is known that the p-points of the elementary abelian p-group E are the elements of $J_E \backslash J_E^2$ [**Ca**]. Several characterizations and a thorough study of p^t-points, as well as $k[A]$-modules of constant p^t-Jordan type for $t \geq 1$ are given in [**Ka**]. Note that the term 'Jordan type' in the references cited above refers to what we call a p-Jordan type. To increase readability, we omit p^i's from our terminology when they are not essential. For instance, we may omit p^r or p^s or both when referring to a p^s-restricted p^t-Jordan type.

When the group is elementary abelian, only p-points which may be p^s-restricted for some $s \geq 1$ exist. The constraints on a restricted Jordan type $\underline{b} \downarrow_{p^s}$ make a restricted $k[E]$-module of constant Jordan type a useful test case for conjectures on $k[E]$-modules of constant Jordan type. In fact, the observation that the constraints on a_i's in many cases are consistent with the conjectures on the Jordan type of a $k[E]$-module of constant Jordan type motivated the definition of restricted modules.

The key result about such constraints is that for an endomorphism Y on M, the decomposition of the operator Y^{p^m} determines that of $Y^{p^{m+s}}$ for $s \geq 1$. More specifically we have the following.

Theorem A. *Suppose that A is an abelian p-group, M is a $k[A]$-module and $\underline{a}$ is a Jordan type for M at a p^s-restricted p^r-point. Then Proposition 3.1 applies to $\underline{a}$. In particular,*

(1) *if $a_i = 0$, then p^s divides the sum $\sum_{j=1}^{p^r} a_j$.*
(2) *if $a_{i-1} = a_{l+1} = 0$, then p^s divides the sum $\sum_{j=i}^{l} a_j$, for $1 < i \leq l < p^r$ and $p^r > 2$.*

There are more conjectures than results on Jordan types of $k[E]$-modules of constant Jordan type as it is a difficult problem even for E of rank 2. For instance, Suslin stated that [**CFP**, Question 9.6] if there are no Jordan blocks of sizes $i-1$ and $i+1$ in the Jordan type of a $k[C_p \times C_p]$-module of constant Jordan type, then there is no Jordan block of size i, for $p > 3$. By Theorem A, for an arbitrary rank elementary abelian p-group E, and a p^s-restricted $k[E]$-module of constant Jordan type if there are no Jordan blocks of sizes $i-1$, $i+1$, for $1 < i < p$ and $p > 3$, then p^s divides the number of Jordan blocks of size i. This suggests the modification in Suslin's conjecture stated below.

Modified Suslin's Conjecture. *If M is a $k[E]$-module of constant Jordan type $\underline{a}$ with $a_{i-1} = a_{i+1} = 0$, then p^s divides a_i for some $s \geq 1$, for $1 < i < p$ and $p > 3$.*

A conjecture by J.Rickard stated in [**Be1**, 4.4] asserts that if there is no Jordan block of size i in the Jordan type of a $k[E]$-module of constant Jordan type, then the total number of Jordan blocks of size at least i is divisible by p. Recently, Benson verified Rickard's conjecture for the special case $i = 1$ [**Be3**, Theorem 1.2]. Even though there is no implication between Suslin's and Rickard's conjectures, Modified Suslin's Conjecture is a special case of Rickard's and Suslin's conjectures. Since our definitions of p^t-Jordan type, constant p^t-Jordan type $k[A]$-module, etc., are for an arbitrary abelian p-group A, it is natural to ask for generalizations of the existing conjectures or state the existing results as conjectures by replacing E with A, and replacing a p-Jordan type with a p^t-Jordan type whenever suitable, for $t \geq 1$. As an example we state a conjecture below generalizing both Suslin's and Rickard's conjectures to $k[A]$-modules of constant p^t-Jordan type and refer to it as *the generalized Suslin and Rickard conjecture.*

Conjecture B. *Suppose that M is a $k[A]$-module of constant p^r-Jordan type $\underline{a}$. Then Proposition 3.1 applies to $\underline{a}$. In particular,*

(1) *if $a_i = 0$, then there is an $s \geq 1$ such that p^s divides the sum $\sum_{j=1}^{p^r} a_j$ for $1 \leq i \leq p^r$.*
(2) *if $a_{i-1} = a_{l+1} = 0$, then there is an $s \geq 1$ such that p^s divides the sum $\sum_{j=i}^{l} a_j$, for $1 < i \leq l < p^r$ and $p^r > 2$.*

Conjecture B is true for p^s-restricted $k[A]$-modules by Theorem B in Section 4.2. Thus for restricted $k[E]$-modules of constant Jordan type Rickard's Conjecture and Modified Suslin's Conjecture are true. In addition, a generalization of [**CFP**, Conjecture 9.5] is true by Corollary 4.8; a generalization of [**CFP**, Conjecture 9.7] is true for $p \geq r$ by Theorem 4.4 (3), see Section 4.3 for further disscussion.

On the other hand, roughly speaking, Theorems 4.1, 4.4, 4.6 are generalizations or variations of results in [**Be**], [**Be2**], [**Be3**]. Thus there are many implications than we include in this article between our results and Bensons'. In particular, Theorem 4.6 (1) generalizes [**Be**, Theorem 1.1]; Theorem 4.4 (3) generalizes [**Be2**, Corollary 1.3]; Corollary 4.8 generalizes [**Be3**, Theorem 1.2]; Theorem 4.1 and Theorem 4.4 (2) generalize [**Be3**, Theorem 1.5]; Theorem 4.4 (2) is a variation of [**Be2**, Theorem 1.2].

Moreover, in [**Be2**] the techniques of [**BP**] is used to obtain constraints on Jordan types with a small non-projective part. In a similar work, Baland [**Ba**] improved some bounds given in [**Be2**] for Jordan types. Our Corollary 4.5 not only agrees with [**Ba**, Theorem 1.2] and has a weaker hypothesis, but also determines the Jordan type completely.

We expect that some of our results for $k[A]$-modules are true if 'resticted' and/or 'divisibility by p' are/is removed, see Conjectures 4.3, 4.7.

A p^t-Jordan type $\underline{b}$ is called *realizable* if there is a $k[A]$-module of constant Jordan type $\underline{b}$ and called *stable* if b_{p^r} is not taken into consideration in which case written as $\underline{b} = (b_1, \ldots, b_{p^t-1}, *)$. A Jordan block of size less than p^t is referred as a *stable Jordan block*. There are some realization results in [**CFS**] and also in [**BP**] giving a method of producing a $k[E]$ module of constant stable Jordan type $(a, 0, \ldots, *)$ from a given vector bundle of rank a.

We noticed that some of the constructions producing a $k[E]$-module of constant Jordan type, such as the W modules described in [**CFP**], can be generalized to $k[A]$-modules of constant p^t-Jordan type. The natural candidates to generalize the constructions or results given for elementary abelian p-groups are homocyclic groups denoted by H, *i.e.*, direct products of C_{p^t}'s for a fixed $t \geq 1$. For instance, if H is of rank 2 and of exponent $p^t > p$, then we define a $k[H]$-module of type W' analogous to the construction of $k[E]$-modules of type W given in [**CFP**], see Definition 3.7. The restrictions of the $k[H]$-module W' to homocyclic subgroups H' of H realizes many Jordan types as shown in Section 3.1.

The outline of the article is as follows. In the current Section 1, we have introduced some definitions, recalled relevant results from the literature and such. Section 2 is devoted to the key lemma of the article. Section 3 includes the key result Theorem A of the article and our theorems about constraints on the Jordan type of a restricted $k[E]$-module as well as a method of producing constant Jordan type modules. In Section 4.1, we consider some special p^t-Jordan types for $k[A]$-modules including the ones inspired by Benson's results on Jordan types for $k[E]$-modules and state new conjectures. In Section 4.2, we discuss Rickard's and Suslin's conjectures. In Section 4.3, we revisit some conjectures and questions stated in Section 9 of [**CFP**].

2. Main Lemma

Let B be a nilpotent $d \times d$ matrix over k with nilpotency at most p^t. The p^t-Jordan type of B is the p^t-tuple $\underline{b}$ with b_i denoting the number of Jordan blocks of size i in the Jordan canonical form of B. Conversely any p^t-tuple is the Jordan type of a nilpotent matrix of nilpotency at most p^t. Naturally, such a matrix B represents a d-dimensional $k[C_{p^t}]$-module M for which the action of the generator of the group C_{p^t} is given by the multiplication with $B + I_d$. Note that, except for the terms on the diagonal, the Jordan canonical form of $B + I_d$ is the same as that of B where I_d is the identity matrix of size d.

Main Lemma. *If B is a $d \times d$ matrix of nilpotency at most p^t having p^t-Jordan type $\underline{b} = (b_1, \dots, b_{p^t})$, then the p^s-restricted p^r-Jordan type $\underline{a} = (a_1, \dots, a_{p^r}) = \underline{b}\downarrow_{p^s}$ of B^{p^s} satifies the following where $r = t - s$.*

(1) $\operatorname{rank}(B^i) = b_{i+1} + 2b_{i+2} + \cdots + (p^t - i)b_{p^t}, \quad 1 \le i < p^t$.

(2) $b_i = \operatorname{rank}(B^{i-1}) - 2\operatorname{rank}(B^i) + \operatorname{rank}(B^{i+1}), \quad 1 < i < p^t$.

(3) $a_i = p^s b_{ip^s} + \Sigma_{j=1}^{p^s-1} j\left[b_{(i-1)p^s+j} + b_{(i+1)p^s-j}\right], \quad 1 \le i \le p^r$, *in particular,*
$a_{p^r} = b_{p^t-p^s+1} + 2b_{p^t-p^s+2} \cdots + (p^s-1)b_{p^t-1} + p^s b_{p^t}$.

(4) $\Sigma_{i=j}^{l} b_i = \operatorname{rank}(B^{j-1}) - \operatorname{rank}(B^j) - \operatorname{rank}(B^l) + \operatorname{rank}(B^{l+1})$, *and*
$$\Sigma_{i=j}^{l} a_i = \operatorname{rank}(B^{(j-1)p^s}) - \operatorname{rank}(B^{jp^s}) - \operatorname{rank}(B^{lp^s}) + \operatorname{rank}(B^{(l+1)p^s})$$
$$= p^s(b_{jp^s} + b_{jp^s+1} + \ldots + b_{lp^s}),\ 1 \le j \le l \le p^r.$$

(5) $\Sigma_{i=1}^{p^t} b_i = d - \operatorname{rank}(B)$, $\Sigma_{i=1}^{p^r} a_i = d - \operatorname{rank}(B^{p^s})$, $d = \Sigma_{i=1}^{p^t} ib_i = \Sigma_{i=1}^{p^r} ia_i$.

(6) *If $a_{i-1} = 0$, then p^s divides the sum $\Sigma_{j=i}^{p^r} a_j$ for $1 < i \le p^r$.*

(7) *If $a_{i-1} = a_{l+1} = 0$, then $\Sigma_{j=i}^{l} a_j = p^s(b_{ip^s} + b_{ip^s+1} + \ldots + b_{lp^s})$ for $1 < i \le l < p^r$. In particular, $a_i = p^s b_{ip^s}$.*

(8) *If $a_{i-2} = a_{i+1} = 0$, $a_{i-1}a_i \ne 0$, then $a_{i-1} + a_i \ge p^s$ for $2 < i \le p^r$.*

(9) *If $a_{i-2} = a_{i+2} = 0$ and $0 < a_j < p$, then $a_{i-1} + a_i + a_{i+1} \ge 2p^s$ for $j = i-1, i, i+1$ and $2 < i < p^r - 1$.*

Proof. (1) Observe that for a Jordan block $[m]$ of size m with eigenvalue 0,
$$\operatorname{rank}([m]^i) = \begin{cases} m - i, & \text{if } i < m; \\ 0, & \text{if } i \ge m. \end{cases}$$

By the hypothesis on B, the Jordan canonical form of B consists of Jordan blocks of sizes at most p^t with eigenvalue 0. Hence, the above formula gives the desired equality

$$\operatorname{rank}(B^i) = \sum_{m=1}^{p^t} b_m(\operatorname{rank}([m]^i)) = \sum_{m=1}^{p^t} b_m(m-i).$$

(2) This follows from the formula in (1).
(3) The conclusions of (1) and (2) are true for the matrix B^{p^s} as well. By (2) we have

$$a_i = \operatorname{rank}(B^{(i-1)p^s}) - 2\operatorname{rank}(B^{ip^s}) + \operatorname{rank}(B^{(i+1)p^s}).$$

The desired formula follows from the computations using (1).
(4) Note that

$$\begin{aligned}
\sum_{i=1}^{p^t} b_i &= \sum_{i=1}^{p^t} \Big(\operatorname{rank}(B^{i-1}) - 2\operatorname{rank}(B^i) + \operatorname{rank}(B^{i+1}) \Big) \\
&= \operatorname{rank}(B^0) - 2\operatorname{rank}(B^1) + \operatorname{rank}(B^2) \\
&+ \operatorname{rank}(B^1) - 2\operatorname{rank}(B^2) + \operatorname{rank}(B^3) \\
&+ \operatorname{rank}(B^2) - 2\operatorname{rank}(B^3) + \operatorname{rank}(B^4) \\
&\quad \vdots \\
&+ \operatorname{rank}(B^{j-1}) - 2\operatorname{rank}(B^j) + \operatorname{rank}(B^{j+1}) \\
&+ \operatorname{rank}(B^{j}) - 2\operatorname{rank}(B^{j+1}) + \operatorname{rank}(B^{j+2}) \\
&\quad \vdots \\
&+ \operatorname{rank}(B^{l-2}) - 2\operatorname{rank}(B^{l-1}) + \operatorname{rank}(B^{l}) \\
&+ \operatorname{rank}(B^{l-1}) - 2\operatorname{rank}(B^{l}) + \operatorname{rank}(B^{l+1}) \\
&\quad \vdots \\
&+ \operatorname{rank}(B^{p^t-3}) - 2\operatorname{rank}(B^{p^t-2}) + \operatorname{rank}(B^{p^t-1}) \\
&+ \operatorname{rank}(B^{p^t-2}) - 2\operatorname{rank}(B^{p^t-1}) \\
&+ \operatorname{rank}(B^{p^t-1}) \\
&= \operatorname{rank}(B^0) - \operatorname{rank}(B) \\
&= d - \operatorname{rank}(B).
\end{aligned}$$

There are many cancellations in the sum $\Sigma_{i=j}^{l} b_i$ as it can be easily seen from the displayed pattern above. What remains is $\operatorname{rank}(B^{j-1}) - \operatorname{rank}(B^j) - \operatorname{rank}(B^l) + \operatorname{rank}(B^{l+1})$ as claimed. Since $\underline{a}$ is p^s-restricted, the formula in (2) for B^{p^s} gives $a_i = \operatorname{rank}(B^{i-1p^s}) - 2\operatorname{rank}(B^{ip^s}) + \operatorname{rank}(B^{(i+1)p^s})$.
(5) The first equality is a special case of part (4) with $l = p^t$. The second one follows also from (4) by replacing B with B^{p^s}. The last one is obvious due to dimensional

reasons.

(6)–(9) By the formula given in (3) each one of $a_1, \dots, a_{p^r-1}$ can be written as a linear combination of $(2p^s-1)$-many b_j's with coefficients in $\{1, \dots, p^s\}$ for suitable j's. In particular, a_{p^r} is a linear combination of only p^s-many b_j's with coefficients $1, \dots, p^s$. When the equations for $a_1, \dots, a_{p^r}$ are listed, the pattern of coefficients becomes noticable,

$$\begin{aligned}
a_{i-1} &= b_{ip^s-2p^s+1} + 2b_{ip^s-2p^s+2} + \ldots + p^s b_{(i-1)p^s} + \\
&\qquad \underline{(p^s-1)b_{ip^s-p^s+1} + \ldots + 2b_{ip^s-p^s+p^s-2} + b_{ip^s-1}}, \\
a_i &= \underline{b_{ip^s-p^s+1} + 2b_{ip^s-p^s+2} + \ldots + (p^s-1)b_{ip^s-1}} + p^s b_{ip^s} + \\
&\qquad \underbrace{(p^s-1)b_{ip^s+1} + \ldots + 2b_{ip^s+p^s-2} + b_{ip^s+p^s-1}} \\
a_{i+1} &= \underbrace{b_{ip^s+1} + 2b_{ip^s+p^s-2} + \ldots + (p^s-1)b_{ip^s+p^s-1}} + p^s b_{(i+1)p^s} + \\
&\qquad (p^s-1)b_{ip^s+p^s+1} + \ldots + 2b_{ip^s+2p^s-2} + b_{ip^s+2p^s-1} \\
&\vdots \\
a_{p^r-1} &= b_{p^t-2p^s+1} + 2b_{p^t-2p^s+2} + \ldots + (p^s-1)b_{p^t-p^s-1} + p^s b_{p^t-p^s} + \\
&\qquad \underbrace{(p^s-1)b_{p^t-p^s+1} + (p^s-2)b_{p^t-p^s+2} + \ldots + b_{p^t-1}} \\
a_{p^r} &= \underbrace{b_{p^t-p^s+1} + 2b_{p^t-p^s+2} + \ldots + (p^s-1)b_{p^t-1}} + p^s b_{p^t}.
\end{aligned}$$

Observe that in $a_1, \dots, a_{p^r}$, the numbers $b_1, \dots, b_{p^s-1}$ appear only in a_1 with respective coefficients $1, \dots, p^s-1$, and b_{ip^s} appears only in a_i with coefficient p^s, for $i = 1, \dots, p^r$. However the remaining b_j's appear twice, one in a_i, the other one in a_{i+1} for some i, with coefficients adding up to p^s. Note also that the sum of the underbraced parts of the equations of the consecutive a_i and a_{i+1} is divisible by p^s, similarly for each consecutive pair, a_{i+1}, a_{i+2} and a_{i+2}, a_{i+3}, up to a_{p^r-1}, a_{p^r}.

Notice further that if $a_{i-1} = 0$, then all of the non-negative integers $b_{ip^s-2p^s+1}$, $b_{ip^s-2p^s+2}, \dots, b_{ip^s-1}$ in its expansion must be zero, that is, the underlined part of the equation for a_i written below must be zero. Hence p^s divides the sum $\Sigma_{j=i}^{p^r} a_j$ proving (6). Similarly, when $a_{l+1} = 0$, p^s divides the sum $\Sigma_{j=l+1}^{p^r} a_j$. Then p^s divides the sum $\Sigma_{j=i}^{l} a_j$ proving (7). Also, if $a_{i-2} = a_{i+1} = 0$ and $a_{i-1}a_i \neq 0$, then $a_{i-1} + a_i \geq p^s$ proving (8). The pattern of the coefficients of b_l's in a_i's implies that if $0 < a_j < p$, for $j = i-1, i, i+1$, and $a_{i-2} = a_{i+2} = 0$, then $a_{i-1} + a_i + a_{i+1} \geq 2p^s$ proving (9).

$\square$

Corollary 2.1. *Let $\underline{b}$ be a p^t-Jordan type with only one non-zero coordinate b_j and $j = p^s k + l$ for some $0 \leq l < p^s$, $0 \leq k \leq p^{t-s}$. Then one of the following holds for the p^s-restricted p^{t-s}-Jordan type $\underline{a} = \underline{b}\downarrow_{p^s}$.*

(1) *If $k = 0$, then $a_1 = (p^s - j)b_j$ is the only non-zero coordinate in $\underline{a}$.*
(2) *If $l = 0$, then $a_k = p^s b_j$ is the only non-zero coordinate in $\underline{a}$.*
(3) *If $kl \neq 0$, then $a_k = (p^s - l)b_j$, and $a_{k+1} = lb_j$ are the only non-zero coordinates in $\underline{a}$.*

Proof. (1) and (2) follow from Main Lemma (3). For item (3), note that if $kl \neq 0$ the number b_j occurs twice as noted in the proof of Main Lemma (6)–(9). Hence there is i such that b_j occurs with coefficients adding up to p^s only in a_i and a_{i+1}, that is $a_i = (p^s - c)b_j$ and $a_{i+1} = cb_j$, for some $0 < c < p^s$. By the hypothesis on $\underline{b}$ there are no other non-zero a_m's, so that $jb_j = ia_i + (i+1)a_{i+1} = (ip^s + c)b_j$. Therefore $j = i(p^s - c) + (i+1)c = ip^s + c = p^s k + l$ implies $i = k$ and $c = l$ proving the statement. □

Recall that the Jordan block $[j]$, equivalenty its p^t-Jordan type $\underline{b}$ with the only non-zero coordinate $b_j = 1$, represents the j-dimensional indecomposable $k[C_{p^t}]$-module for $j \leq p^t$. The p^s-restricted p^{t-s}-Jordan type $\underline{b}\downarrow_{p^s}$ represents the restriction of the module to the subgroup $C_{p^{t-s}}$. The Jordan type of $[j]^{p^s}$ can be computed by Corollary 2.1 from that of $[j]$ since the Jordan type of $[j]$ has only one non-zero coordinate. In addition, although not so practical, it may as well be computed from a chart with a nice pattern which is demonstrated in the following example.

Example 2.2. To display the pattern of the coefficients of b_j's in a_i's we take a 5^2-Jordan type $\underline{b}$ and let $\underline{a}$ be its 5-restricted 5-Jordan type. Then

$$\begin{aligned}
a_1 &= b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5 + 4b_6 + 3b_7 + 2b_8 + b_9,\\
a_2 &= b_6 + 2b_7 + 3b_8 + 4b_9 + 5b_{10} + 4b_{11} + 3b_{12} + 2b_{13} + b_{14},\\
a_3 &= b_{11} + 2b_{12} + 3b_{13} + 4b_{14} + 5b_{15} + 4b_{16} + 3b_{17} + 2b_{18} + b_{19},\\
a_4 &= b_{16} + 2b_{17} + 3b_{18} + 4b_{19} + 5b_{20} + 4b_{21} + 3b_{22} + 2b_{23} + b_{24},\\
a_5 &= b_{21} + 2b_{22} + 3b_{23} + 4b_{24} + 5b_{25}.
\end{aligned}$$

Let $C_{25} = \langle g \rangle$, and set $J = \mathrm{Rad}(k[C_{25}])$, $J_{\langle g^5 \rangle} = \mathrm{Rad}(k[\langle g^5 \rangle])$. Note that J^i is represented by the Jordan block $[25 - i]$. Furthermore, let $\underline{b}$ be the 5^2-Jordan type of J. Then $\underline{a} = \underline{b}\downarrow_5$ is the Jordan type of $J_{\langle g^5\rangle}$. The decomposition of the restriction $J^i\downarrow_{\langle g^5\rangle}$ in terms of the indecomposable $k[\langle g^5\rangle]$-modules $J^j_{\langle g^5\rangle}$, $j = 1, \ldots, 5$, is determined by the 5-restricted 5-Jordan type $\underline{a}$ for $i = 1, \ldots, 25$. The Jordan type $\underline{a}$ can also be determined from the chart having a significant pattern as displayed below.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
k	J^{23}	J^{22}	J^{21}	J^{20}	J^{19}	J^{18}	J^{17}	J^{16}	J^{15}	J^{14}	J^{13}	J^{12}	J^{11}	J^{10}	J^9	J^8	J^7	J^6	J^5	J^4	J^3	J^2	J	J^0
1	2	3	4	5	4	3	2	1																
					1	2	3	4	5	4	3	2	1											
										1	2	3	4	5	4	3	2	1						
															1	2	3	4	5	4	3	2	1	
																				1	2	3	4	5

In the above table, the top row lists the dimension $25 - i$ of J^i. Except the first four, the entries below the double line in each column add up to 5. The rows below the double line give the number of indecomposable summands in the decomposition of $J^i\downarrow_{\langle g^5\rangle}$. More precisely, the integer in the t-th row below the double line, under J^i, is the number of t-dimensional indecomposable module $J^{5-t}_{\langle g^5\rangle}$ in the decomposition of $J^i\downarrow_{\langle g^5\rangle}$. There is one type of indecomposable summand in $J^i\downarrow_{\langle g^5\rangle}$ for $i = 20, \ldots, 24$ and also for i divisible by 5. For the remaining i's there are exactly two different indecomposable summands having consecutive dimensions and multiplicities adding up to 5. For instance, the 16 dimensional J^9 has 5^2-Jordan type $\underline{b} = (0, \ldots, 0, b_{16} = 1, 0, \ldots, 0)$. Thus its restriction $J^9\downarrow_{\langle g^5\rangle}$ is of constant Jordan type $\underline{a} = (a_1, \ldots, a_5) = (0, 0, 4, 1, 0)$, equivalently, $J^9\downarrow_{\langle g^5\rangle}$ decomposes as follows

$$J^9\downarrow_{\langle g^5\rangle} \cong (J^2_{\langle g^5\rangle})^{\oplus 4} \oplus J_{\langle g^5\rangle}.$$

Similarly for the 5^2-Jordan type $\underline{b} = (0, \ldots, 0, b_{18}, 0, \ldots, 0)$ with $b_{18} = 1$, the restricted Jordan type is $\underline{a} = (0, 0, 2, 3, 0)$.

3. Main Theorem, Realization of some Jordan types

As noted before, results for matrices are results for modules as well. Main Lemma can be restated in terms of modules as follows.

Proposition 3.1. *Let A be an abelian p-group of exponent $p^t > p$, M be a $k[A]$-module, $r = t - s$ and M^{1+w} be the fixed point set of $1 + w$ in M for $w \in J_A$. If $\underline{b}$ and $\underline{a}$ are the p^t and p^r-Jordan types of $M\downarrow_{\langle 1+z\rangle}$, $M\downarrow_{\langle 1+z^{p^s}\rangle}$, respectively, for some p^t-point z in J_G, then for $1 \le i < p^{t-s}$ the following equalities hold;*

$$
\begin{aligned}
a_i &= p^s b_{ip^s} + \Sigma_{j=1}^{p^r-1} j\left[b_{(i-1)p^s+j} + b_{(i+1)p^s-j}\right],\\
a_{p^r} &= b_{p^t-p^s+1} + 2b_{p^t-p^s+2}\cdots + (p^s-1)b_{p^t-1} + p^s b_{p^t},\\
\Sigma_{i=j}^{l} b_i &= \dim(z^{j-1}M) - \dim(z^jM) - \dim(z^lM) + \dim(z^{l+1}M),\\
\Sigma_{i=j}^{l} a_i &= \dim(z^{(j-1)p^s}M) - \dim(z^{jp^s}M) - \dim(z^{lp^s}M) + \dim(z^{(l+1)p^s}M)\\
&= p^s(b_{jp^s} + b_{jp^s+1} + \ldots + b_{lp^s}),\ \textit{in addition,}\\
\dim(M^{1+z}) &= \Sigma_{i=1}^{p^t} b_i \ \textit{and}\ \dim(M) = \Sigma_{i=1}^{p^t} ib_i = \Sigma_{i=1}^{p^r} ia_i.
\end{aligned}
$$

Furthermore,

(1) *if* $a_i = 0$*, then* p^s *divides the sum* $\Sigma_{j=i+1}^{p^r} a_j (= \dim(z^{ip}M) - \dim(z^{(i+1)p}M))$.
(2) *if* $a_{i-1} = a_{l+1} = 0$*, then* $\Sigma_{j=i}^{l} a_j = p^s(b_{ip^s} + b_{ip^s+1} + \ldots + b_{lp^s})$
for $1 < i \leq l < p^r$*, in particular,* $a_i = p^s b_{ip^s}$.
(3) *if* $a_{i-2} = a_{i+1} = 0$*,* $a_{i-1}a_i \neq 0$*, then* $a_{i-1} + a_i \geq p^s$ *for* $2 < i \leq p^r$.
(4) *if* $a_{i-2} = a_{i+2} = 0$*, and* $0 < a_j < p$*, for* $j = i-1, i, i+1$*, then*
$a_{i-1} + a_i + a_{i+1} \geq 2p^s$ *for* $2 < i < p^r - 1$.
(5) *if* $b_{ip^s+1} = b_{ip^s+2} = \cdots = b_{ip^s+p^s-1} = 0$*, then* $a_{i+1} = a_i = 0$.
(6) *if* $b_{ip^s+1} = b_{ip^s+2} = \cdots = b_{ip^s+2p^s-1} = 1$*, then* $a_{i+1} = p^s$.
(7) *if* $\underline{a} = (\ldots, 0, a, \ldots, a, 0, \ldots)$ *with* n*-many* a*'s,* $n \geq 1$*, then* p *divides* a *or* n.

Proof. Let B be the matrix representing the action of z and $\underline{b}$ be the Jordan type of B. Then B^{p^s} represents the action of z^{p^s} and has Jordan type $\underline{a}$ which is $\underline{b}\downarrow_{p^s}$. All the statements are basically restatements of the items of Main Lemma in Section 2 for B and/or B^{p^s}. Note that $wm = 0$ if and only if $(1+w)m = m$ for $m \in M$ and $w \in J_A$. Thus $\ker(w)$ on M is the same as the fixed point set of $1+w$ in M. Hence, the equalities in the non-itemized part follow directly from Main Lemma (1)–(5) and the fact that $\dim(M) = \dim(M^{1+w}) + \dim(wM)$.
The parts (1), (2), (3), (4) follow from the parts of Main Lemma (6), (7), (8), (9) respectively.
(5) and (6) follow from direct computations using the formula for a_i's in terms of b_j's and (7) follows from (2).

□

Remark 3.2. Recall from the proof of Main Lemma (6)–(9) that for $j > p^s$ b_j occurs in two consecutive a_i's with coeficients adding up to p^s. Since 2 divides p^s only for the prime $p = 2$, the coefficient of b_j in a_i and a_{i+1} cannot be the same if $p \neq 2$. The coefficient of $b_{i2^s+2^{s-1}}$ in a_i, also in a_{i+1}, is 2^{s-1}. For instance, for the 2^4-Jordan type $\underline{b}$ and its 2^2-restricted 2^2-Jordan type we know that the coefficient of b_6 is 2 in a_1 and a_2, similarly the coefficient of b_{10} is 2 in a_2 and a_3.

The following is a key result of the article, hence stated as a theorem even though it is a corollary of Proposition 3.1.

Theorem A. *Suppose that A is an abelian p-group, M is a $k[A]$-module and $\underline{a}$ is a Jordan type for M at a p^s-restricted p^r-point. Then Proposition 3.1 applies to $\underline{a}$. In particular,*

(1) *if $a_i = 0$, then p^s divides the sum $\sum_{j=1}^{p^r} a_j$.*
(2) *if $a_{i-1} = a_{l+1} = 0$, then p^s divides the sum $\sum_{j=i}^{l} a_j$, for $1 < i \leq l < p^r$ and $p^r > 2$.*

Proof. Since $\underline{a}$ is p^s-restricted there is a Jordan type $\underline{b}$ such that $\underline{a} = \underline{b}\downarrow_{p^s}$ for some $s \geq 1$. Thus the statements (1) and (2) follow from Proposition 3.1 (1) and (2), respectively.

□

3.1. CONSTRUCTING MODULES OF CONSTANT JORDAN TYPE

A p^t-Jordan type $\underline{b}$ is called *realizable* or *realized by M* if there is a $k[A]$-module M of constant p^t-Jordan type $\underline{b}$. We give examples of $k[A]$-modules of constant Jordan type realizing many special Jordan types.

Remark 3.3. There is no implication between constant p^t-Jordan types for various t. Namely, it can happen that different p^t-Jordan types can restrict to the same p^l-Jordan type for $1 \leq l \leq t$. Hence constant p^l-Jordan type modules need not be of constant p^t-Jordan type for $1 \leq l < t$. On the other hand, if the group A has a generator of order p^l, then A has a non-restricted p^l-point. Hence M is of constant p^t-Jordan type does not imply M is of constant p^l-Jordan type for $1 \leq l < t$, see [**Ka**, Example 4.3, and 4.4].

Theorem 3.4. *Suppose that A is a an abelian p-group with a subgroup A' of exponent at least p^t, and M is a $k[A]$-module of constant p^t-Jordan type $\underline{b}$. Then*

(1) *$M\downarrow_{A'}$ is of constant p^t-Jordan type $\underline{b}$.*
(2) *if M is of constant p^l-Jordan type $\underline{c}$, then $\underline{b}\downarrow_{p^{t-l+q}} = \underline{c}\downarrow_{p^q}$ for $0 \leq q \leq l \leq t$, regarding p^0-restricted as no restriction.*

Proof. (1) By the hypothesis, the only p^t-Jordan type for M is $\underline{b}$. Since A' is a subgroup of A, every p^t-point of A' is necessarily a p^t-point of A. Hence there is only one p^t-Jordan type for $M\downarrow_{A'}$ proving that it is of constant Jordan type.

(2) Since $\underline{c}$ is the only p^l-Jordan type, and $\underline{b}\downarrow_{p^{t-l}}$ is a p^{t-l}-restricted $p^{t-(t-l)}$-Jordan type they are equal. Note that $\underline{b}\downarrow_{p^{m+n}}= (\underline{b}\downarrow_{p^m})\downarrow_{p^n}$. Hence any further p^q-restriction of the two Jordan types must be equal for $0 \leq q \leq l \leq t$.

□

By the above theorem, many p^r-Jordan types are realizable, to realize more Jordan types we will make use of some constructions in the case of elementary abelian groups. The natural candidates to generalize the constructions or results given for elementary abelian p-groups are homocyclic groups H, *i.e.*, direct products of C_{p^t}'s for a fixed $t \geq 1$. The next theorem in the case of an elementary abelian group E is given in [**CFP**, Example 2.2].

Theorem 3.5. *The p^t-Jordan type $(r-1, 1, 0, \ldots, 0)$ is realized by the $k[H]$-module $k[H]/J^2$ of constant p^t-Jordan type where H is homocyclic abelian p-group of exponent p^t, and of rank r.*

Proof. As a vector space $k[H]$ can be written as $k \oplus k(h_1 - 1) \oplus \ldots \oplus k(h_r - 1) \oplus J^2$ where $\{h_1, \ldots, h_r\}$ is a generating set for H, the dimension of $(h_i - 1)k[H]/J^2$ is 1. Hence $xk[H]/J^2$ is of dimension 1 and $x^2 k[H]/J^2 = 0$ for $x \in J \backslash J^2$. Thus the p^t-Jordan type is $(r-1, 1, 0, \ldots, 0)$ by Main Lemma (2).

□

Note that, for $A' \leq A$, the n-th syzygy $\Omega^n_{A'}(k)$ of the trivial $k[A]$-module k is isomorphic to the non-free part of the restriction $\Omega^n_A(k)\downarrow_{A'}$.

Theorem 3.6. *Suppose that A is an abelian p-group with a subgroup A' of exponent at least p^t and $1 < l \leq t$. Then $\Omega^n_A(k)$ is of constant p^l-Jordan type, and its restriction $\Omega^n_A(k)\downarrow_{A'}$ is of constant p^l-Jordan type $(0, \ldots, 0, 1, *)$, when n is odd; is of constant the p^t-Jordan type $(1, 0, \ldots, 0, *)$, when n is even for $p^t > 2$; and $(1, *)$ for all n for $p^t = 2$.*

Proof. The non-free part of the restriction of $\Omega^n_A(k)$ to $k[C_{p^t}]$ is isomorphic to $\Omega^n_{C_{p^t}}(k)$ and $\Omega^n_{C_{p^t}}(k)$ is isomorphic to $J_{C_{p^t}}$ for odd n and isomorphic to k for even n. Hence the stable p^t-Jordan type of $\Omega^n_A(k)$ is $(0, \ldots, 0, 1, *)$ for odd n; $(1, \ldots, 0, *)$ for even n and $p^t > 2$. Since $\Omega^n_{C_2}(k) \cong k$ for all n, $(1, *)$ is the stable 2-Jordan type for $\Omega^n_A(k)$ when $p^t = 2$. Since A' is of exponent at least p^t, there are p^t-points of A', and p^t-Jordan type $\underline{b}$ of $\Omega^n_A(k)\downarrow_{A'}$ at a p^t-point is as stated above.

□

3.2. MODULES OF TYPE W'

When E is of rank 2, $k[E]$-modules of type W, also referred as W modules, are introduced in [**CFS**] as examples of $k[E]$-modules of constant Jordan type. For a homocyclic p-group H of rank 2, and exponent bigger than p, we first define a $k[H]$-module of type W' by extending the construction of the W modules given in [**CFS**] from E to H. The W' modules and their restrictions are of constant Jordan type. Hence they realize many Jordan types, see Theorem 3.8. The W modules, hence W' modules, can be represented by diagrams in the shape of the letter W.

Definition 3.7. Let $n \geq d \geq 1$ and $p^t \geq d$. For the group $C_{p^t}\times C_{p^t}$ with generators g, h, let $x = g-1$, $y = h-1$, and define $W'_{n,d}$ as the $k[C_{p^t}\times C_{p^t}]$-module having a generating set $\{v_1, \ldots, v_n\}$ and relations generated by

$$xv_1 = 0 = yv_n; \quad x^d v_i = 0 = yv_i - xv_{i+1}, \quad \text{for} \quad 1 \leq i \leq n-1.$$

For $1 \leq n \leq d$, we set $W'_{n,d}$ as the $W'_{n,n}$ above.

Theorem 3.8. *Suppose that $n \geq d \geq p^s > l \geq 0$ are integers such that $d = p^s i + l$ for some $i \geq 1$ and $p^t \geq d$, $p^t > p$. Then ${W'}_{n,d}$ is a $k[C_{p^t}\times C_{p^t}]$-module of constant p^t-Jordan type*

$$\underline{b} = (1, \ldots, 1, b_d, 0 \ldots, 0) \textit{ where } b_d = n - d + 1,$$

and the restriction of $W'_{n,d}$ to $k[C_{p^{t-s}} \times C_{p^{t-s}}]$ is of constant p^{t-s}-Jordan type

$$\begin{aligned} \underline{a} = \underline{b}{\downarrow_{p^s}} &= (p^{2s}, \ldots, p^{2s}, a_i, a_{i+1}, 0, \ldots, 0) \;\textit{ with} \\ a_i &= \frac{1}{2}(p^{2s} - p^s - l(l-1)) + (p^s - l)b_d, \;\textit{ and} \\ a_{i+1} &= \frac{1}{2}l(l-1) + lb_d. \end{aligned}$$

In particular, ${W'}_{p^t,p^t}$ has p^t-Jordan type $\underline{b} = (1, \ldots, 1)$ and its p^s-restriction ${W'}_{p^t,p^t}\downarrow_{C_{p^{t-s}}\times C_{p^{t-s}}}$ is of constant p^{t-s}-Jordan type

$$\underline{a} = (p^{2s}, \ldots, p^{2s}, \tfrac{1}{2}p^s(p^s+1)).$$

Proof. Let $H = C_{p^t}\times C_{p^t}$ with generators g, h, and let $x = g-1$, $y = h-1$. An element z in J_H is a p^t-point if $z = ax+by+w$ for $0 \neq (a,b) \in k^2$ and some w in J_H^2 [**Ka**]. As in the proof of the analogous result given for $C_p\times C_p$ in [**CFS**, Proposition 2.3] we observe that the k-subspace spanned by $(ax+by)v_1, \ldots, (ax+by)v_n$ is the same as the one spanned by $yv_1, \ldots, yv_{n-1}$, for $0 \neq (a,b) \in k^2$. Hence $W'_{n,d}$ is of constant p^t-Jordan type. Since $yv_n = xv_1 = 0$, we have $y^2 v_{n-1} = xyv_n = 0$. Thus we obtain that $\dim(y^i M) - \dim(y^{i+1}M) = n - i$ for all $i = 1, \ldots, d-1$. Therefore,

$b_i = \dim(y^{i-1}M) - 2\dim(y^iM) + \dim(y^{i+1}M) = 1$, for $i = 1, \ldots, d-1$. Since $x^dM = 0$, $y^dm = 0$, $b_d = \dim(y^{d-1}M) = n-d+1$ and $b_i = 0$ for $i = d+1, \ldots, p^t$ *i.e.*, $\underline{\mathrm{b}} = (1, \ldots, 1, b_d, 0 \ldots, 0)$ with $b_d = n-d+1$ as claimed. Then $\underline{\mathrm{a}} = \underline{\mathrm{b}}\downarrow_{p^s}$ is computed by the formulas in Proposition 3.1 for a_i's.

□

Remark 3.9. Many p^t-Jordan types are realizable by the $k[H]$-modules of type W' and their restrictions to subgroups of H. In addition, candidates for realizable p^t-Jordan types can be chosen using Proposition 3.1.

4. Conjectures on Jordan types and Special Cases

In this section we give necessary conditions on the Jordan type of a restricted $k[A]$-module of constant p^r-Jordan type $\underline{\mathrm{a}}$, p^s-restricted from a p^t-Jordan type $\underline{\mathrm{b}}$, by exploiting the equality $a_i = p^s b_{ip^s} + \Sigma_{j=1}^{p^{t-s}-1} j\left[b_{(i-1)p^s+j} + b_{(i+1)p^s-j}\right]$ thoroughly as we did up to this point. Some of these necessary conditions may still be necessary for not necessarily restricted modules. Hence by removing the hypothesis 'restricted' on the modules we can obtain many conjectures such as Conjectures 4.3 and 4.7.

It should be noted here that when dealing with the Jordan types of restricted $k[E]$-modules, as in Section 4.2 and Section 4.3 in the verification of the conjectures by Rickard and Suslin, it is sufficient to consider p-restricted p-Jordan types, that is, restrictions from C_{p^2} to C_p. For a group element g in G of order p^t, the order of $g^{p^{t-2}}$ is p^2 and hence $g^{p^{t-1}} - 1$ is a restricted p-point of the maximal elementary abelian subgroup E of G.

4.1. SPECIAL JORDAN TYPES FOR $k[E]$-MODULES

We focus on various special p^t-Jordan types analogous to the p-Jordan types considered for the case the group is elementary abelian in [**Be**], [**Be2**], [**Ba**] et al. These include the cases of p^t-Jordan types for which the multiplicity of the Jordan blocks is not divisible by p^s for some $s \geq 1$ or the Jordan blocks of only even or only odd sizes occur or there are exacly two Jordan blocks or there are Jordan blocks of consecutive sizes, etc. Firstly, we elaborate on the following theorem by Benson.

Theorem [**Be3**, 1.5]**.** *Suppose that M is a $k[E]$-module of constant Jordan type with all stable Jordan blocks of distinct odd size or all of distinct even size. Then* *the stable Jordan type is either of the form* $(1, 0, \dots, 0, *)$ *or* $(0, \dots, 0, 1, *)$ *or has at least four Jordan blocks.*

We have the following generalization of Theorem [**Be3**, 1.5] for restricted $k[A]$-modules without the assumption that the Jordan blocks are of distinct sizes.

Theorem 4.1. *Suppose that* M *is a* p^s*-restricted* $k[E]$*-module of constant Jordan type with all stable Jordan blocks of only odd size or of only even size. Then the Jordan type of* M *is of the form* $(p^s t_1 + q, 0, p^s t_3, 0, p^s t_5, 0, \dots, 0, p^s t_{p^t})$ *for some integer* $q \geq 0$ *or* $(0, p^s t_2, 0, p^s t_4, 0, \dots, 0, a, a')$ *where* t_i*,* a*,* a' *are non-negative integers with* $a + a'$ *divisible by* p^s*.*

Proof. Substituting 0's in the equation for a_i given in Proposition 3.1 yield the desired Jordan types. More precisely, if only odd or only even size Jordan blocks exist, then $a_{i-1} = a_{i+1} = 0$ with p^s dividing a_i, or $a_i = a_{i+2} = 0$ with p^s dividing a_{i+1}. In particular, if $a_2 = 0$, then q is given by $a_1 = b_1 + 2b_2 + \dots + p^s b_{p^s} = q + p^s b_{p^s}$. □

The following corollary is a variation of [**Be3**, Theorem 1.5].

Corollary 4.2. *Suppose* M *is a restricted* $k[E]$*-module of constant stable Jordan type with the number of Jordan blocks of the same size not divisible by* p*. If all stable Jordan blocks are of only odd size or of only even size, then the Jordan type of* M *is of the form* $(a, 0, \dots, 0, *)$ *or* $(0, \dots, 0, a, *)$*, for some integer* $a \geq 0$*.*

A reasonable conjecture is obtained by removing the hypothesis 'restricted' in the statement of Theorem 4.1.

Conjecture 4.3. *Suppose that* M *is a* $k[A]$*-module of constant* p^t*-Jordan type with all stable Jordan blocks of only odd size or of only even size. Then the Jordan type of* M *is of the form* $(p^s t_1 + r, 0, p^s t_3, 0, p^s t_5, 0, \dots, 0, p^s t_{p^t})$ *for some integer* $r \geq 0$ *or* $(0, p^s t_2, 0, p^s t_4, 0, \dots, 0, a, a')$ *where* t_i*,* a*,* a' *are non-negative integers with* $a + a'$ *divisible by* p^s*.*

Next, we focus on Jordan types with small multiplicities as in many results of Benson.

Theorem 4.4. *Suppose that M is a restricted $k[E]$-module of constant Jordan type $\underline{a}$ and $S_{i,l} = a_i + \ldots + a_l$, for $1 \leq i \leq l \leq p$.*

(1) *If p does not divide $S_{i,l}$, for $1 < i < l < p$, then*
$a_{i-1}a_i \neq 0$ with $a_{i-1} + a_i \geq p$ or $a_l a_{l+1} \neq 0$ with $a_l + a_{l+1} \geq p$.
(2) *If $S_{i,l} < p$, then $a_{i+1} = \ldots = a_{l-1} = 0$, for $1 \leq i \leq l-2$, $l \leq p$.*
(3) *If $0 < S_{i,l} < p$, then $S_{i-1,l+1} \geq p$, for $1 < i \leq l < p$. In particular, if $a_i = a$, $a_{i+1} = 1$, $a_{i+2} = 0$ for some $1 \leq i < p-1$, then $a \geq p-1$.*

Proof. By the definition of a restricted $k[E]$-module, there is an $s \geq 1$ and a p^{s+1}-Jordan type $\underline{b}$ for M such that $\underline{a} = \underline{b}\downarrow_{p^s}$. Also $S_{i,l} = a_i + \ldots + a_l$ for $1 \leq i \leq l \leq p$.
(1) follows from Proposition 3.1 (2).
(2) Let $a_k \neq 0$ for some $i+1 \leq k \leq l-1$, then $0 < a_k \leq S_{i,l} < p$ and by Proposition 3.1 (2), $a_{k-1} \neq 0$ and $a_{k-1} + a_k \geq p$, or $a_{k+1} \neq 0$ and $a_k + a_{k+1} \geq p$. Since $i < i+1 \leq k \leq l-1 < l$, we have $i \leq k-1 < k+1 \leq l$. Thus $S_{i,l} \geq a_{k-1} + a_k + a_{k+1} \geq p$ contradicting the assumption $S_{i,l} < p$. Therefore, $a_{i+1} = \ldots = a_{l-1} = 0$.
(3) By the previous part (2) $a_{i+1} = 0$, $a_{l-1} = 0$. Hence $0 < S_{i,l} = a_i + a_l < p$. Again by Proposition 3.1 (2), if $a_i \neq 0$, then $a_{i-1} \neq 0$ with $a_{i-1} + a_i \geq p$; if $a_l \neq 0$, then $a_{l+1} \neq 0$ with $a_l + a_{l+1} \geq p$. In any case $S_{i-1,l+1} \geq a_{i-1} + a_i + a_l + a_{l+1} \geq p$. Applying this to the particular case with $S_{i+1,i+1} = 1$ gives $S_{i,i+2} = 1 + a \geq p$ as claimed. □

For restricted $k[E]$-modules, Theorem 4.4 (2) generalizes [**Be3**, Theorem 1.5]. Because, for the Jordan types in the hypothesis of [**Be3**, Theorem 1.5], $S_{1,p} < \frac{p+1}{2} < p$. Hence, by Theorem 4.4 (2), we have $a_2 = \ldots = a_{p-1} = 0$. In the case that there are distinct even length Jordan types, $a_1 = 0$ as well. Then $a_{p-1} = 1$ is the only non zero term except for a_p. A similar argument works for distinct odd length Jordan types.

All of our results are independent of the rank of the group unlike the following [**Be2**, Theorem 1.2], and its Corollary [**Be2**, 1.3].

Theorem [**Be2**, 1.2]**.** *If M is a $k[E]$-module of stable constant Jordan type $\underline{a}$ with $\Sigma_{j=1}^{p-1} ja_j \leq \min(r-1, p-2)$ where r is the rank of E, then $a_i = 0$ for $2 \leq i \leq p-1$.*

Corollary [**Be2**, 1.3]**.** *If M is a $k[E]$-module of stable constant Jordan type $(a, 1, 0 \ldots, 0, *)$ with $p > r$ and r is the rank of E, then $a \geq r-2$.*

Note that Theorem 4.4 (3) verifies Conjecture[**CFP**, 9.7], see section 4.3; Theorem 4.4 (3) showing $a \geq r-1$, for $p \geq r$ is stronger than [**Be2**, Corollary 1.3] showing $a \geq r-2$, for $p > r$.

In addition, if $\sum_{j=1}^{p-1} ja_j \leq \min\{r-1, p-2\}$ and $p \geq r$, then $S_{1,p-1} \leq \sum_{j=1}^{p-1} ja_j < p$. Hence Theorem 4.4 (2) implying that $a_i = 0$, for $2 \leq i \leq p-2$ is a variation of [**Be2**, Theorem 1.2], only $i = p-1$ is missed out.

Baland [**Ba**, 1.2, 1.3] used a variation of the technique used in the proof of [**Be2**, Theorem 1.2] to obtain restrictions on the sizes of the Jordan blocks when there are only two Jordan blocks with the additional assumption on the rank of the group. We should point out that the Jordan type in the hypothesis of [**Ba**, Theorem 1.3] consisting of exactly 2 non-free, not one dimensional Jordan blocks of the same size cannot be realized by a restricted $k[E]$-module since $S_{1,p-1} = 2 < p$, $a_2 = \ldots a_{p-1} = 0$ by Theorem 4.4 (2), or by Proposition 3.1 (8). For a restricted $k[E]$-module, Corollary 4.5 below not only agrees with [**Ba**, Theorem 1.2] but also determines the Jordan type completely and has a weaker hypothesis. It is a variation of [**Be2**, Theorem 1.2] and [**Be3**, Theorem 1.5].

Corollary 4.5. *If M is a restricted $k[E]$-module of constant Jordan type with exactly two non-free Jordan blocks of distinct sizes i, j with $1 \leq i < j < p$, $p > 3$, then $i = 1$, $j = p-1$ and p divides (1+ the number of Jordan blocks of size p).*

Proof. By the hypothesis $S_{1,p-1} = 2 < p$. Then $a_2 = \ldots = a_{p-2} = 0$ by Theorem 4.4 (2). Since there are exactly two distinct size Jordan blocks, it is necessary that $a_1 = a_{p-1} = 1$ as claimed.

□

In the next theorem we focus on some special Jordan types with small multiplicities similar to the ones considered in the related articles mentioned babove.

Theorem 4.6. *Suppose that M is a restricted $k[E]$-module of constant Jordan type $\underline{a}$ containing Jordan blocks of consecutive sizes with multiplicities smaller than p.*

(1) *If $\underline{a}$ is of the form $(\ldots, 0, a, 0, \ldots)$, then $a = 0$.*
(2) *If $\underline{a}$ is of the form $(\ldots, 0, a, b, 0, \ldots)$, then $b = p - a$, and $a \neq b$.*
(3) *If $\underline{a}$ is of the form $(\ldots, 0, a, b, \ldots)$ or $(\ldots, b, a, 0, \ldots)$, then $a \geq p-b$. Moreover, if $b \neq 0$, then $a \neq 0$.*

(4) *If* $\underline{a}$ *is of the form* $(\ldots, 0, a, b, c, 0, \ldots)$ *with* $abc \neq 0$*, then* $a + b + c \geq 2p$*.*
(5) *If* $\underline{a}$ *is of the form* $(\ldots, 0, 2, b, c, 0, \ldots)$ *with* $c \neq 0$*, then* $b = c = p - 1$*.*
(6) *If* $\underline{a}$ *is of the form* $(\ldots, 0, 1, b, c, 0, \ldots)$*, then* $b = p - 1$ *and* $c = 0$*.*
(7) *If* $\underline{a}$ *is of the form* $(\ldots, a, b, c, 0, \ldots)$ *or* $(\ldots, 0, c, b, a, \ldots)$ *with* a*,* b*,* c *not all zero, then*
if $p \not| b + c$ *then* $a \neq 0$*; if* $a = 0$ *then* $bc \neq 0$*; if* $b = 0$*, then* $c = 0$*.*
(8) *If* $\underline{a}$ *is of the form* $(\ldots, 0, a, \ldots, a, 0, \ldots)$ *with* n*-many* a*'s, then* $a = 0$*, or* p *divides* n*.*

Proof. Since M is a restricted $k[E]$-module, there is an $s \geq 1$ and a p^{s+1}-Jordan type $\underline{b}$ such that $\underline{a} = \underline{b}\downarrow_{p^s}$. Hence, we can use the results given so far for restricted Jordan types, especially Proposition 3.1 (2).
(1) By Proposition 3.1 (2), p divides a. Hence the hypothesis $a < p$ implies $a = 0$.
(2) By Proposition 3.1 (2), p divides $a + b$. Moreover, since $0 < a < p$ and $b < p$, we have $0 < a + b < 2p$. Hence $b = p - a$ as claimed. There are at least four coordinates in the given Jordan type, hence $p > 3$. If $a = b$, then $p = 2a$ which is not possible. Therefore, $a \neq b$.
(3) and (4) are immediate from Proposition 3.1 (3).
(5) Since there is a 0 preceeding 2, either $2 = 2b_n$ or $2 = b_m$ for some m and n with $b_n = 1$. If $2 = b_m$, then $b = (p-1)2+q$ for some $q \geq 0$ contradicting the hypothesis that $b < p$. If $2 = 2b_n$, then $b = (p - 2)b_n + q = p - 2 + q$ for some $0 \leq q \leq 1$. Then the hypothesis $b < p$ implies that $b = p - 1$. Hence, by Proposition 3.1 (2) $2 + p - 1 + c = p + 1 + c = pl$ for some $l \geq 2$. Since $c < p$, we have $pl < 2p + 1$. Thus $l = 2$ and hence $c = p - 1$.
(6) There is $j > 1$ with $a_j = 1$ and $a_{j+1} = b$. Then from the equations listed at the begining of the proof of Main Lemma (6)–(9) we see that $p \leq 1 + b$, *i.e.*, $p - 1 \leq b$. By the hypothesis we have $p - 1 \leq b < p$, hence $b = p - 1$. By Proposition 3.1 (2), p divides $p + c$. The hypothesis on $c < p$ implies $p + c < 2p$, therefore $c = 0$.
(7) This can be verified by Proposition 3.1 (1)–(3) using similar arguments as in the previous parts, and (8) Follows from Proposition 3.1 (8).

□

Similar to Theorem 4.4 (3), but with the additional assumption $a < p$, Theorem 4.6 (3) is stronger than Benson's [**Be2**, Corollary 1.3].

It is reasonable to expect our results to hold for arbitrary $k[E]$-modules.

Conjecture 4.7. *Theorem 4.4, Theorem 4.6 are true without the hypothesis 're-stricted' on the* $k[E]$*-module.*

4.2. GENERALIZATION OF CONJECTURES BY RICKARD AND SUSLIN

Suslin and Rickard'sonjectures on Jordan types are already stated in the introduction. Since we are generalizing these and state similar conjectures in this section we restate them for the convenience of the reader.

Rickard's Conjecture [**Be1**, 4.4]**.** *If M is a $k[E]$-module of constant Jordan type $\underline{a}$ with $a_i = 0$, then p divides the total number of Jordan blocks of size bigger than i, for $1 \leq i < p$.*

The case $r = 2$ of the following conjecture first appeared in [**CFP**, Question 9.6].

Suslin's Conjecture [**Be1**, 4.5]**.** *If M is a $k[E]$-module of constant Jordan type $\underline{a}$ with $a_{i-1} = a_{i+1} = 0$, then $a_i = 0$, for $2 < i < p$, and $p > 3$.*

Since we know by Proposition 3.1 (2) that for a restricted $k[E]$-module of constant Jordan type with no Jordan blocks of sizes $i-1$, $i+1$, p divides the number of Jordan blocks of size i, for $1 < i < p$ and $p > 3$. Thus we modify Suslin's conjecture.

Modified Suslin's Conjecture. *If M is a $k[E]$-module of constant Jordan type $\underline{a}$ with $a_{i-1} = a_{i+1} = 0$, then p^s divides a_i for some $s \geq 1$, for $2 < i < p$, and $p > 3$.*

It is not known if there is a (p^s-restricted) $k[E]$-module of constant Jordan type $(\ldots, 0, a, 0, \ldots)$ with $a \neq 0$ and divisible by p^s for some $s \geq 1$.

Note that Suslin's and Rickard's conjectures imply Modified Suslin's conjecture. We generalize Suslin's and Rickard's conjectures to $k[G]$-modules.

Conjecture B. *Suppose that M is a $k[A]$-module of constant p^r-Jordan type $\underline{a}$. Then Proposition 3.1 applies to $\underline{a}$. In particular,*

(1) *if $a_i = 0$, then there is an $s \geq 1$ such that p^s divides the sum $\sum_{j=1}^{p^r} a_j$ for $1 \leq i \leq p^r$.*
(2) *if $a_{i-1} = a_{l+1} = 0$, then there is an $s \geq 1$ such that p^s divides the sum $\sum_{j=i}^{l} a_j$, for $1 < i \leq l < p^r$ and $p^r > 2$.*

Theorem A has the following corollary which verifies Conjecture B.

Theorem B. *Conjecture B is true for a p^s-restricted $k[A]$-module of constant p^t-Jordan type $\underline{a}$. In particular, Rickard's Conjecture and Modified Suslin's Conjecture are true for restricted $k[E]$-modules of constant Jordan type $\underline{a}$.*

Proof. Let M be a p^s-restricted $k[A]$-module of constant p^r-Jordan type $\underline{a}$. Then there is a p^{t+s}-Jordan type $\underline{b}$ of M such that $\underline{a} = \underline{b}\downarrow_{p^s}$. Then we can use Theorem A with $\underline{a}$ to verify Conjecture B, in particular, part (1) implies Rickard's conjecture, part (2) implies Modified Suslin's Conjecture.

□

For restricted $k[E]$-modules, Theorem B is stronger than [**Be3**, Theorem 1.2] which proved the special case $i = 1$ of Rickard's conjecture.

Corollary 4.8. *If M is a p^s-restricted $k[A]$-module of constant p^t-Jordan type $\underline{a}$, then $\underline{a}$ satisfies the conditions given in Proposition 3.1, in particular, for the restricted Jordan type of the form $\underline{a} = (\ldots, 0, a, 0, \ldots, 0, a')$ with $p^s \mid a$ and $p^s \mid a'$.*

For restricted $k[E]$-modules, Corollary 4.8 verifies a strengthened form of [**CFP**, Conjecture 9.5], and it is stronger than [**Be**, Theorem 1.1]. It is mainly due to Proposition 3.1 (7), see Section 4.3.

4.3. CONJECTURES AND QUESTIONS FROM [**CFP**]

We revisit below some questions and conjectures stated in in [**CFP**, Section 9] on $k[E]$-module of constant Jordan type briefly. Recall that Suslin's conjecture first appeared in [**CFP**, 9.6]. We will use our notation for Jordan types rather than that of [**CFP**].

Question[**CFP**, 9.1]**.** *For a given Γ what Jordan types are realized as the Jordan type of a finite dimensional $k[\Gamma]$-module M with constant Jordan type?*

See Theorems 3.5, 3.6, 3.8 for examples of Γ and Jordan types that are realized.

Conjecture[**CFP**, 9.5]**.** *Let E be rank ≥ 2, with $p > 3$. Then there does not exist a finite dimensional $k[E]$-module of constant stable Jordan type $(0, 1, 0, \ldots, 0, *)$.*

Conjecture 9.5 is true for restricted $k[E]$-modules regardless of the rank of E, by Corollary 4.8, and also by Theorem 4.6 part (1), or part (8). In fact, Corollary 4.8, Theorem 4.6 (1), (8) are generalizations of [**Be**, Thorem 1.1] which rules out the existence of a $k[E]$-module of constant stable Jordan type $(\ldots, 0, 1, 0, \ldots)$.

Conjecture[**CFP**, 9.7]**.** *Let E be of rank $r \geq 2$, with $p > 3$. Then there does not exist a finite dimensional $k[E]$-module of constant stable Jordan type $(a, 1, 0, \ldots, 0, *)$ with $r - 2 \geq a$.*

Equivalently, for $r \geq 2$, $p > 3$, if there is a $k[E]$-module of constant Jordan type $(a, 1, 0, \ldots, 0, *)$, then $a \geq r - 1$.

Conjecture 9.7 is true by Theorem 4.4 (3) for restricted $k[E]$-modules whenever $p \geq r$. Because, the equality $a_2 = S_{2,2} = 1 < p$ implies $S_{1,3} = 1 + a \geq p \geq r$ by Theorem 4.4 (3), so that $a \geq r - 1$ for $p \geq r$. Also Theorem 4.6 (3) verifies Conjecture 9.7 for $a < p$, and $p \geq r$. In fact, our results are for more general Jordan types, namely, $(\ldots, a, 1, 0, \ldots)$. A weaker similar result is [**Be2**, Corollary 1.3] of [**Be2**, Theorem 1.2] which proves $a \geq r - 2$ for $p > r$.

References

[**Ba**] S. Baland, *Modules of constant Jordan type with two non-projective blocks*, J. Algebra **346** (2011), 343-350

[**Be**] D. Benson, *Modules of constant Jordan type with one non-projective block* Algebr. Represent. Theory **13** (2010), 315-318.

[**Be1**] D. Benson, *Modules for elementary abelian p-groups*, Proceedings of the International Congress of Mathematicians 2010 (ICM 2010), 113–124

[**Be2**] D. Benson, *Modules of constant Jordan type with small non-projective part*, Algebr. Represent. Theory 16 **1**, (2013), 29-33.

[**Be3**] D. Benson, *Modules of constant Jordan type and a conjecture of Rickard*, J. Algebra, to appear.

[**BP**] D. Benson, J. Pevtsova, *A realization theorem for modules of constant Jordan type and vector bundles*, Trans. AMS. to appear.

[**Ca**] J. Carlson, *The varieties and the cohomology ring of a module*, J. Algebra, **85** (1983), 104–143.

[**CFP**] J. F. Carlson, E. Friedlander, J. Pevtsova, *Modules of constant Jordan type*, J. Reine Angew. Math. **614** (2008), 191–234.

[**CFP1**] J. F. Carlson, E. Friedlander, J. Pevtsova, *Representations of elementary abelian p-groups and bundles on Grassmannians*, Adv. Math. **229**, no.5, (2012), 2985–3051.

[**CFS**] J. F. Carlson, E. Friedlander, A. Suslin, *Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$*, Comment. Math. Helv. **86** (2011), 609–657.

[**FP**] E. Friedlander, J. Pevtsova, *Representation theoretic support spaces for finite group schemes*, Amer. J. Math. **127** (2005), 379–420, with Erratum in Amer. J. Math. **128** (2006), 1067–1068.

[**Ka**] S. Ö. Kaptanoğlu, *p-Power points and modules of constant p-power Jordan type*, Communications in Algebra, **39**, (2011), 3781–3800.

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