STRUCTURE AND DETECTION THEOREMS FOR $k[C_2 \times C_4]$-MODULES

SEMRA ÖZTÜRK KAPTANOĞLU

Abstract. Let $k[G]$ be the group algebra, where G is a finite abelian p-group and k is a field of characteristic p. A complete classification of finitely generated $k[G]$-modules is available only when G is cyclic, C_{p^n}, or $C_2 \times C_2$. Tackling the first interesting case, namely modules over $k[C_2 \times C_4]$, some structure theorems revealing the differences between elementary and non-elementary abelian group cases are obtained. The shifted cyclic subgroups of $k[C_2 \times C_4]$ are characterized. Using the direct sum decompositions of the restrictions of a $k[C_2 \times C_2]$-module M to shifted cyclic subgroups we define the set of multiplicities of M. It is an invariant richer than the rank variety. Certain types of $k[C_2 \times C_4]$-modules having the same rank variety as $k[C_2 \times C_2]$-modules can be detected by the set of multiplicities, where $C_2 \times C_2$ is the unique maximal elementary abelian subgroup of $C_2 \times C_4$.

1. Introduction

Let M be a finitely generated $k[G]$-module, where G is a finite group of order divisible by p and k is a field of characteristic $p > 0$. The cyclic group of order n is denoted by C_n. When M is considered as a module over a subalgebra $k[A]$ of $k[G]$ for a subgroup A of the group of units of $k[G]$, we write $M{\downarrow_A}$, and refer to it as the *restriction* of M to $k[A]$ (or A) or simply as the *restricted module* . Some properties of a $k[G]$-module M, such as complexity, are detected by $M{\downarrow_E}$ as E runs through elementary abelian subgroups of G; see [**AE**], [**Kr**]. Theorems of this nature are referred to as *detection theorems*.

The rather rich theory for modules over an elementary abelian p-group E is not of much use when the group is non-elementary abelian. An essential motivation for this work was to clearify the reasons for that by presenting a detailed study of modules over the smallest non-elementary abelian 2-group $C_2 \times C_4$ via their restrictions to various subalgebras of $k[C_2 \times C_4]$ and $k[C_2 \times C_2]$ where $C_2 \times C_2$ is

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the unique maximal elementary abelian subgroup of $C_2 \times C_4$. Given the fact that $k[C_2 \times C_4]$ is of wild representation type [**Be1**, p. 114] what we provide in this article as structure theorems is essential for studying modules over it. It should be noted here that the only abelian non-cyclic p-group whose indecomposable modules are classified is the group $C_2 \times C_2$; [**Ba**], [**HeRe**], [**Co**]. There are infinitely many non-isomorphic indecomposable $k[C_2 \times C_2]$-modules.

The work of Carlson defining the rank variety for a module over an elementary abelian p-group is the main source of inspiration for this study [**Ca**]. However, what we achieve is the result of our new approach; namely, rather than just focusing on whether the restrictions of a $k[E]$-module are free or not as Carlson does, we further take into account the direct sum decomposition of the restrictions of a $k[C_2 \times C_4]$-module to preserve more information so that the module may be characterized by that information. The direct sum decompositions of the restrictrictions of a $k\mathcal{G}$-module at p-points, roughly speaking order p-subgroups, where $\mathcal{G}$ is a finite group scheme are studied througly by Friedlander, Suslin and Pevtsova in [**FP**], [**FP1**], [**SFP**]. Examples in an earlier version of this work [**Ka1**] were mentioned in [**SFP**]. They consider only cyclic subgroups of order p whereas we consider cyclic subgroups of order p^n, $n \geq 1$. As a result, we can distinguish some modules which are not possible to distinguish by considering only order p subgroups.

The restrictions we consider are $M\!\downarrow_{\langle 1+x\rangle}$ for x in the Jacobson radical J_G, or simply J, of the group algebra $k[G]$. The structure theorems, Theorems 4.1, 4.3, 4.5, reveal the structure of the restrictions $M\!\downarrow_{\langle 1+x\rangle}$ for various x. They provide a good insight why modules over elementary abelian groups behave better than modules over non-elementary abelian groups, and indicate what type of changes in the hypotheses lead to similar results when the group is non-elementary abelian. One consequence is the characterization of the shifted cyclic subgroups of $k[C_2 \times C_4]$, see (7).

By the formula provided in Corollary 2.2, it is not difficult to compute the multiplicities of i-dimensional indecomposable summands of $M\!\downarrow_{\langle 1+x\rangle}$. Hence the direct sum decomposition of $M\!\downarrow_{\langle 1+x\rangle}$ can be determined for any x in J without much difficulty. This leads us to define a new computable invariant, $\mathcal{N}_G(M)$, for a $k[G]$-module M called the *set of multiplicities* of M where G is an abelian p-group see (3). This definition becomes very useful when the group is $C_2 \times C_2$ since the domain of definition, J/J^2, coincides with that of the rank variety, see Definition 1.5. Recovering the rank variety from the set of multiplicities is very easy.

Even though it is not discussed in this paper, we can state that when G is $C_2 \times C_2 \times C_2$ or $C_3 \times C_3$ the domain of definition of $\mathcal{N}_G(M)$ becomes J/J^3 rather than J/J^2. That also justifies our choice of groups for this study.

Our detection theorems, Theorems 5.1 and 5.3, are applications of this invariant. They show that certain types of modules can be identified by their set of multiplicities. There is a geometric interpretation of the multiplicities of the indecomposable summands of $M{\downarrow_C}$ when M is a "realizable" $k[G]$-module and C is a cyclic 2-subgroup of a group G [**Ka**].

In order to state our main results we need to introduce our notation and definitions some of which are similar to the ones in [**Ca**]. In doing this we also recall some results from the literature to put our results into the proper context. Recall that the notions of projectivity, injectivity and freeness coincide for modules over $k[G]$ when G is a p-group and k is a field of characteristic p.

The key result concerning the restricted modules $M{\downarrow_{\langle 1+x\rangle}}$ is Dade's Lemma [**Da**, Lemma 11.8] when the group E is an elementary abelian p-group.

Dade's Lemma 1.1. *Suppose E is an elementary abelian p-group of order p^n generated by $e_1, \ldots, e_n$, k is an algebraicaly closed field of caharacteristic p, and M is a finitely generated $k[E]$-module. Then, M is free if and only if $M{\downarrow_{\langle 1+x\rangle}}$ is free for all x of the form $\alpha_1(e_1-1)+\ldots\alpha_n(e_n-1)$ where $(\alpha_1,\ldots,\alpha_n)\in k^n$.*

Another result in these lines is the following (Lemma 6.4 in [**Ca**]).

Theorem 1.2. *Let M be a $k[E]$-module, and x, y in $J\backslash J^2$ such that $x \equiv y \pmod{J^2}$. Then $M{\downarrow_{\langle 1+x\rangle}}$ is free if and only if $M{\downarrow_{\langle 1+y\rangle}}$ is free.*

Definition 1.3. A (cyclic) subgroup S of the units of $k[G]$ is called a *shifted (cyclic) subgroup* of $k[G]$ whenever $k[G]{\downarrow_S}$ is free as a $k[S]$-module.

Dade's Lemma allowed Carlson to define the shifted cyclic subgroups for an elementary abelian p-group E. In our notation they are the subgroups $\langle 1+x\rangle$ of

$k[E]$, where x is in $J\backslash J^2$. Dade's Lemma together with Theorem 1.2 made the definition of $V_E^r(M)$, *the rank variety of* M well defined. It consists of points $\overline{x}$ in J/J^2 for which $M\!\downarrow_{\langle 1+x\rangle}$ is not free together with the point zero.

In order to describe the shifted cyclic subgroups of $k[C_2 \times C_4]$ explicitly, we need to introduce notation for the generators of the group $C_2 \times C_4$.

Notation 1.4. Except in Section 2, in the rest of this article, G denotes the group $C_2 \times C_4$ generated by e, f of orders 2 and 4, respectively; E denotes its unique elementary abelian subgroup $C_2 \times C_2$. Consider the ideals $\mathbb{J}^{(2)} \supset \mathbb{J}^{(3)}$ of kG contained in J^2 so that

$$\begin{aligned} J/\mathbb{J}^{(2)} &\cong k\overline{(e-1)} \oplus k\overline{(f-1)} \oplus k\overline{(f-1)^2} \cong k^3, \\ J/\mathbb{J}^{(3)} &\cong k\overline{(e-1)} \oplus k\overline{(f-1)} \oplus k\overline{(f-1)^2} \oplus k\overline{(f-1)^3} \cong k^4. \end{aligned} \tag{1}$$

We omit the bars to simplify the notation. When $M\!\downarrow_{\langle 1+x\rangle}$ and $M\!\downarrow_{\langle 1+y\rangle}$ have the same indecomposable summands (up to isomorphism) together with the same multiplicities, that is, when x and y have the same Jordan canonical form, we write

$$M\!\downarrow_{\langle 1+x\rangle} \equiv M\!\downarrow_{\langle 1+y\rangle}$$

for x, y in J provided that $k\langle 1+x\rangle$ and $k\langle 1+y\rangle$ are isomorphic subalgebras.

Our main structure theorems are as folows.

Theorem 4.1. (i) *If $x \in J$, then $k[G]\!\downarrow_{\langle 1+x\rangle}$ is free if and only if $x \notin \mathbb{J}^{(2)}$.*

(ii) *If M is a $k[G]$-module, then M is free if and only if the restriction $M\!\downarrow_{\langle 1+x\rangle}$ is free for all x in $J\backslash\mathbb{J}^{(2)}$.*

The first part of Theorem 4.1 is the analogue of [**Ca**, Lemma 6.1]. By that the shifted cyclic subgroups of $k[G]$ can be written in the form $\langle 1+x\rangle$ for any $\overline{x}$ in $J/\mathbb{J}^{(2)}$, see Remark 4.2. The second part of Theorem 4.1 is the counterpart of Dade's Lemma.

Theorem 4.3. *Let M be a $k[G]$-module, x, y in $J\backslash\mathbb{J}^{(2)}$, and $x \equiv y \pmod{\mathbb{J}^{(2)}}$.*

(i) *If x^2 is not zero, then $M\!\downarrow_{\langle 1+x\rangle} \equiv M\!\downarrow_{\langle 1+y\rangle}$.*

(ii) *$M\!\downarrow_{\langle 1+x\rangle}$ is free if and only if $M\!\downarrow_{\langle 1+y\rangle}$ is free.*

The second part of the above theorem implies that the freeness of $M\!\downarrow_{\langle 1+x\rangle}$ is well defined modulo $\mathbb{J}^{(2)}$. In other words, $J/\mathbb{J}^{(2)}$ for G is the analogue of the J/J^2 for the elementary abelian groups. In the terminology of [**FP**] this amounts to showing that $\langle 1+x\rangle$ and $\langle 1+y\rangle$ are in the same equivalence class as $|\langle 1+x\rangle|$-points of $k[G]$. Moreover, by Theorem 4.5 the Jordan form of x is well defined modulo $\mathbb{J}^{(3)}$.

Theorem4.5. *Let M be a $k[G]$-module, x, y be in $J\backslash J^2$, and $x \equiv y \pmod{\mathbb{J}^{(3)}}$. Then*

$$M\!\downarrow_{\langle 1+x\rangle} \equiv M\!\downarrow_{\langle 1+y\rangle} .$$

The analogue of Theorem 4.5 for the group E is given below.

Theorem 4.7. *Let M be a $k[E]$-module and let x, y be in $J\backslash J^2$ such that $x \equiv y$ $\pmod{J^2}$. Then*

$$M\!\downarrow_{\langle 1+x\rangle} \equiv M\!\downarrow_{\langle 1+y\rangle} .$$

Due to the fact that all indecomposable $k[C_2]$-modules, up to isomorphism, are k and $k[C_2]$, $M\!\downarrow_{\langle 1+x\rangle} \cong (k)^{\eta_1(x)} \oplus (k[\langle 1+x\rangle])^{\eta_2(x)}$ when $\langle 1+x\rangle$ is isomorphic to C_2.

Thus the pair $\eta(x) = (\eta_1(x), \eta_2(x))$ consisting of the multiplicities of the indecomposable summands describes the restricted module $M\!\downarrow_{\langle 1+x\rangle}$ up to isomorphism completely and $\eta_i(x)$'s are easy to compute by the formula given in Corollary 2.2(i).

Definition 1.5. For a $k[E]$-module M, the *set of multiplicities* is defined as

$$\mathcal{N}_E(M) = \Big\{ \big[x;\eta(x)\big] \bigm| \overline{x} \in J/J^2\backslash 0 \Big\},$$

it is well defined by Theorem 4.7.

The rank variety $V_E^r(M)$ can be recovered from the set of multiplicities $\mathcal{N}_E(M)$ easily, see Remark 4.4.

Definition 1.6. A $k[H]$-module M is called an *isotypical* $k[H]$*-module (of type N)* whenever M is isomorphic to m copies of an indecomposable $k[H]$-module N for some $m \geq 1$.

Theorem 5.1. *If M is a finitely generated isotypical $k[E]$-module, then $\mathcal{N}_E(M)$ determines M completely (up to isomorphism) except that isotypical modules of type $\Omega^n(k)$ and of type $\Omega^{-n}(k)$ can not be distinguished, where $\Omega^n(k)$ denotes the n-th Heller shift (or n-th syzygy) of the trivial module k.*

Following the definition and notation given in [**Be2**, p. 190], for ζ in $H^l(H;k)$, L_ζ denotes the kernel of the $k[H]$-homomorphism that represents the image of ζ under the isomorphism $H^l(H;k) \cong \mathrm{Hom}_{k[H]}(\Omega^l(k),k)$; see (4). We write $\mathbb{L}_\zeta$ instead of L_ζ when the group is G.

We set the cohomology algebras as follows:

$$\begin{aligned} H^*(G;k) &= k[t_1,\tau]\otimes\Lambda(v), &&\text{and} \quad \zeta_\alpha = \alpha_2 t_1^2 + \alpha_1\tau, \\ H^*(E;k) &= k[t_1,t_2], &&\text{and} \quad \xi_\alpha = \alpha_2 t_1 + \alpha_1 t_2, \end{aligned} \tag{2}$$

where $\alpha = (\alpha_1,\alpha_2)$ is in $k^2\backslash\{(0,0)\}$, and t_1, t_2, v are of degree 1, and τ is of degree 2. It can be shown that the rank varieties of the modules $\mathbb{L}_{\zeta_\alpha}$ and of the induced modules $L_{\xi_\alpha}\!\uparrow_E^G$ are the same; namely, the line through α and the origin. However, restrictions of these modules to the shifted cyclic subgroup $\langle u_\alpha\rangle$ of $k[E]$ corresponding to the point α are have different decompositions. Thus these modules can be distinguished by their sets of multiplicities when considered as modules over E, see (9) for the computation of $\mathcal{N}_E(L_{\xi_\alpha^n})$.

Theorem 5.3. *If M is a finitely generated isotypical $k[G]$-module of type $\mathbb{L}_{\zeta_\alpha^n}$ or induced from an isotypical $k[E]$-module of type $L_{\xi_\alpha^n}$, then M is completely determined (up to isomorphism) by its set of multiplicities $\mathcal{N}_E(M\!\downarrow_E)$ when considered as a $k[E]$-module.*

As the last theorem of the introduction we state a well known theorem due to Chouinard [**Ch**] used several times in the article.

Theorem 1.7. (Chouinard) *Let H be a finite group, M be a $k[H]$-module. Then M is a free $k[H]$-module if and only if $M\!\downarrow_A$ is a free $k[A]$-module for all elementary abelian p-subgroups of H.*

The proofs of our theorems are self-contained and use linear algebra methods and basic homological algebra techniques. Section 2 is devoted to results that are valid for any abelian p-group. Thereafter G is the group $C_2 \times C_4$. Section 3 consists of preliminary lemmas for $k[C_2\times C_4]$-modules. Section 4 contains the structure theorems. Section 5 is devoted to examples and applications of our multiplicities set which are referred as detection theorems.

2. General Results on Restrictions of Modules

In this section, G denotes an abelian p-group. When G is the cyclic p-group of order p^n, the list of indecomposable $k[G]$-modules (up to isomorphism) is given by the i-dimensional vector spaces which are ideals of $k[G]$, namely, the J^{p^n-i}, for i in $\{1,\ldots,p^n\}$. Hence for any abelian p-group G, a finitely generated $k[G]$-module M, and x in $J_G \backslash J_G^2$ with $k\langle 1+x\rangle \cong k[C_{p^n}]$ we have the direct sum decomposition

$$M{\downarrow_{\langle 1+x\rangle}} \cong (k)^{\eta_1(x)} \oplus (J^{p^n-2})^{\eta_2(x)} \oplus \cdots \oplus (J)^{\eta_{p^n-1}(x)} \oplus (k\langle 1+x\rangle)^{\eta_{p^n}(x)}$$

where $\eta_i(x)$, or simply η_i, denotes the multiplicity of the i-dimensional indecomposable summands of $M{\downarrow_{\langle 1+x\rangle}}$, and $J = J_{\langle 1+x\rangle}$. Thus the module $M{\downarrow_{\langle 1+x\rangle}}$ can be represented by $\eta(x) = (\eta_1(x),\ldots,\eta_{p^n}(x))$. These $\eta_i(x)$'s are easy to compute by our formula given in Corollary 2.2, however it is not clear how to find the suitable ideal, say I_G, such that, congruence modulo I_G implies the equality $\eta(x)=\eta(y)$ for x and y in J. If such an ideal is determined, then the set of multiplicities $\mathcal{N}_G(M)$ will be well-defined over J/I_G, where

$$\mathcal{N}_G(M) = \Big\{ \big[x;\eta(x)\big] \bigm| \overline{x} \in J/I_G \backslash 0 \Big\}. \tag{3}$$

For $\mathcal{N}_{C_2\times C_4}(M)$ see Remark 4.6(2).

Lemma 2.1. *Let X be a $d\times d$ nilpotent matrix over a field $\mathbb{F}$ and η_t denote the number of $t\times t$ Jordan blocks in the Jordan form of X. Then*

(i) $\eta_t = \operatorname{rank}(X^{t-1}) - 2\operatorname{rank}(X^t) + \operatorname{rank}(X^{t+1})$ *for* $t \geq 1$;

(ii) *the number of Jordan blocks in X of dimension less than or equal to t is*

$$\operatorname{rank}(X^0) - \operatorname{rank}(X) - \operatorname{rank}(X^t) + \operatorname{rank}(X^{t+1}),$$

the number of those of dimension greater than t is $\operatorname{rank}(X^t) - \operatorname{rank}(X^{t+1})$.

The proof of Lemma 2.1 depends on the fact that

$$\operatorname{rank}([j_t]^r) = \begin{cases} t-r, & \text{if } r<t,\\ 0, & \text{if } r\geq t,\end{cases}$$

where $[j_t]$ denotes the $t\times t$ Jordan matrix belonging to the eigenvalue zero. The following are immediate corollaries of Lemma 2.1.

Corollary 2.2. *Let G be an abelian group whose order is divisible by p, and M be a $k[G]$-module. If x is in J and $\langle 1+x\rangle$ is a cyclic subgroup of $k[G]$ such that $k\langle 1+x\rangle$ is isomorphic to $k[C_{p^s}]$, then*

(i) $M\downarrow_{\langle 1+x\rangle} \cong (k)^{\eta_1} \oplus (J^{p^s-2})^{\eta_2} \oplus (J^{p^s-3})^{\eta_3} \oplus \cdots \oplus (J^2)^{\eta_{p^s-2}} \oplus (J)^{\eta_{p^s-1}} \oplus (k[C_{p^s}])^{\eta_{p^s}}$, *where*

$$\eta_i = \dim(x^{i-1}M) - 2\dim(x^i M) + \dim(x^{i+1}M);$$

(ii) *the number of indecomposable summands of $M\downarrow_{\langle 1+x\rangle}$ of dimension less than or equal to t is* $\dim(M) - \dim(xM) - \dim(x^t M) + \dim(x^{t+1}M)$, *the number of those of dimension greater than t is* $\dim(x^t M) - \dim(x^{t+1}M)$. *In particular, the number of non-free indecomposable summands of $M\downarrow_{\langle 1+x\rangle}$ is equal to* $\dim(M) - \dim(xM) - \dim(x^{p^s-1}M) = \dim(M^{\langle 1+x\rangle}) - \dim(x^{p^s-1}M)$.

(iii) $M\downarrow_{\langle 1+x\rangle}$ *is free if and only if* $\dim(xM) = \frac{p^s-1}{p^s}\dim(M)$ *if and only if* $\dim(x^{p^s-1}M) = \frac{\dim(M)}{p^s}$ *if and only if* $\dim(xM) + \dim(x^{p^s-1}M) = \dim(M)$.

Corollary 2.3. *If G is an abelian group whose order is divisible by p, M is a $k[G]$-module, and x, y are in J, then the following are equivalent.*

(i) *The restrictions $M\downarrow_{\langle 1+x\rangle}$ and $M\downarrow_{\langle 1+y\rangle}$ have the same decomposition.*

(ii) *For all i, $\eta_i(x)$ and $\eta_i(y)$ are the same.*

(iii) *For all i, $\dim(x^i M)$ and $\dim(y^i M)$ are the same.*

Remark 2.4. Note that the number of non-free indecomposable summands of $M\downarrow_{\langle 1+x\rangle}$ is equal to the dimension of the Tate cohomology group

$$\hat{H}^0(\langle 1+x\rangle; M\downarrow_{\langle 1+x\rangle}).$$

Let $\langle u\rangle$ be a shifted cyclic subgroup of G and let ζ be in $H^n(G;k)$. Denote the image of ζ under the restriction map $\mathrm{res}^G_{\langle u\rangle}$ by $\zeta\downarrow_{\langle u\rangle}$. When ζ is zero, L_ζ is defined as $\Omega^n(k)\oplus\Omega^1(k)$; otherwise L_ζ is the kernel of a $k[G]$-homomorphism $\hat{\zeta}$ representing ζ, i.e. it fits in the following short exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^n(k) \xrightarrow{\hat{\zeta}} k \longrightarrow 0. \tag{4}$$

For more information on Ω^n, Ω^{-n}, and L_ζ, we refer to [**Be2**, p. 190]. We write $\Omega^n_G(k)$ if the group needs to be indicated.

Lemma 2.5. *Let G be an abelian p-group, $\langle u\rangle$ be a shifted cyclic subgroup of $k[G]$ and ζ be in $H^n(G;k)\backslash 0$. Then $\Omega^m_{\langle u\rangle} \cong \Omega^i_{\langle u\rangle}(k)$ where i is 0, or 1 if m is even, or odd respectively. The following isomorphisms hold:*

(i) $\Omega^n_G(k)\downarrow_{\langle u\rangle}\cong \Omega^n_{\langle u\rangle}(k) \oplus (k\langle u\rangle)^s$ *for some s in $\mathbb{N}$. The same is true if n is replaced by $-n$.*

(ii) *For some s and l in $\mathbb{N}$,*

$$(L_\zeta)\downarrow_{\langle u\rangle}\cong \begin{cases} L_{\zeta\downarrow_{\langle u\rangle}} \oplus (k\langle u\rangle)^s, & \text{if } \zeta\downarrow_{\langle u\rangle}\neq 0, \\ \Omega^n_{\langle u\rangle}(k) \oplus \Omega^1_{\langle u\rangle}(k) \oplus (k\langle u\rangle)^l, & \text{if } \zeta\downarrow_{\langle u\rangle}= 0. \end{cases}$$

Proof. (i) follows from the definitions of $\Omega^n(k)$, $\Omega^{-n}(k)$, and shifted cyclic subgroups, and the property that a short exact sequence remains short exact when restricted to $k\langle u\rangle$. (ii) follows from the definition of L_ζ and part (i). □

3. Restrictions of $k[C_2 \times C_4]$-Modules

Here we return to the group $G = C_2 \times C_4$. This section is devoted to answers to various questions that naturally arise in studying a $k[G]$-module through its restricitions to shifted cyclic subgroups. In the rest of this section, $\mathcal{B}$ is the following basis

(5) $\mathcal{B} = \{1, e-1, f-1, f^2-1, (f-1)^3, (e-1)(f-1), (e-1)(f-1)^2, (e-1)(f-1)^3\}$.

Some properties of the nilpotent elements of the group algebra $k[G]$ are given in Lemma 3.1, Theorem 3.2, and Lemma 3.3.

The *norm element* ν_H of a group algebra $k[H]$ is the sum of all elements of H. When H is a cyclic group generated by h of order p^n the norm element $\nu_H = \sum_{g\in H} g$ can also be expressed as $(h-1)^{p^n-1}$.

Lemma 3.1. *Let x, y be in $J\backslash 0$ and $x \equiv y \pmod{J^2}$. Then the following hold.*

(i) $x^4 = 0$, *and* $x^2 = a(f^2-1)$ *for some a in k.*

(ii) *If x is in J^2, then $x^2 = 0$.*

(iii) *If x is in J^3, then $xJ^2 = 0$.*

(iv) $x^3 = 0$ *if and only if* $x^2 = 0$.

(v) $x^2 = 0$ *if and only if* $\langle 1+x\rangle \cong C_2(\leq k[G])$.

(vi) $x^2 \neq 0$ *if and only if* $\langle 1+x\rangle \cong C_4(\leq k[G])$.

(vii) $x^2 \neq 0$ *if and only if* $k[\langle 1+x\rangle] \cong k[C_4]$.

(viii) $k\langle 1+x\rangle$ *is a direct summand of* $k[G]{\downarrow}_{\langle 1+x\rangle}$.

(ix) $x^2 = y^2$, $x^3 = 0$ *iff* $y^3 = 0$, $\langle 1+x\rangle \cong \langle 1+y\rangle$, *and* $k\langle 1+x\rangle \cong k\langle 1+y\rangle$.

Proof. Write x and y using the basis $\mathcal{B}$ given in (5) as

$$x = a(e-1) + b(f-1) + c(f-1)^2 + r_1\nu_{<f>} + r_2(e-1)(f-1) + r_3\nu_E + r_4\nu_G$$

(6)

$$y = a(e-1) + b(f-1) + c(f-1)^2 + s_1\nu_{<f>} + s_2(e-1)(f-1) + s_3\nu_E + s_4\nu_G$$

for some a, b, c, r_i, s_i in k. Then we have

$$x^2 = b^2(f^2-1) \quad \text{and} \quad x^3 = ab^2\nu_E + b^3\nu_{<f>} + r_2b^2\nu_G.$$

It is clear that the coefficient b plays a significant role. Proofs of (i)–(ix) are now easy verifications. □

The restricted module $k[G]{\downarrow}_{\langle f^2\rangle}$ is free as $\langle f^2\rangle$ is a subgroup of G. However, $\overline{(f-1)^2}$ is zero in J/J^2. Thus it is clear that we need a substitute for J/J^2 to extend Dade's Lemma. By Lemma 3.1 the non-zero elements of J are of two types depending on whether their square is zero or not. The set of elements of J whose square is zero is denoted by $\mathbb{J}_E$, that is, $\mathbb{J}_E = J_E \oplus \mathbb{J}^{(2)} = k(e-1)\oplus k(f^2-1)\oplus \mathbb{J}^{(2)}$, where $\mathbb{J}^{(2)}$ is as given in (1).

Theorem 3.2. *Let x, y be in $\mathbb{J}_E\backslash\mathbb{J}^{(2)}$ such that $x \equiv y \pmod{\mathbb{J}^{(2)}}$. There is a unit $u = 1+n$ with n in $\mathbb{J}_E\backslash\mathbb{J}^{(2)}$ such that if $E_x = \langle u, 1+x\rangle$ and $E_y = \langle u, 1+y\rangle$, then*

(i) *the groups E_x, E_y are shifted subgroups of $k[G]$ which are isomorphic to E,*

(ii) *the group algebras $k[E]$, $k[E_x]$, $k[E_y]$ are isomorphic subalgebras of $k[G]$.*

Proof. (i) By hypothesis, $\overline{x} = x + \mathbb{J}^{(2)}$ is non-zero. Then there is an n in $\mathbb{J}_E$ such that $\{\overline{x}, \overline{n}\}$ is a basis for $\mathbb{J}_E/\mathbb{J}^{(2)}$. We can write x and y as in (6) with $b = 0$. Let

$$n = n_e(e-1) + n_{f^2}(f-1)^2 + n_1(f-1)^3 + n_2(e-1)(f-1) + n_3\nu_E + n_4\nu_G,$$

where n_e, n_{f^2}, n_i are in k. By hypothesis, (a, c) and (n_e, n_{f^2}) are not equal to $(0,0)$. Moreover, x and n are k-linearly independent modulo $\mathbb{J}^{(2)}$. This is true if and only if $an_{f^2} - cn_e$ is non-zero. Since the field is of characteristic 2, $an_{f^2} + cn_e = an_{f^2} - cn_e$ is non-zero. Hence

$$nx = (an_{f^2} + cn_e)\Big(1 + \frac{an_1 + cn_2 + r_1n_e + r_2n_{f^2}}{an_{f^2} + cn_e}(f-1)\Big)\nu_E$$

is not zero. Set

$$v_x = 1 + \frac{an_1 + cn_2 + r_1 n_e + r_2 n_{f^2}}{an_{f^2} + cn_e}(f-1).$$

It is clear that v_x is a unit in $k[G]$. Since n is in $\mathbb{J}^{(2)}$, we have $n^2 = 0$ by Lemma 3.1 (ii). Therefore $u = 1 + n$ is a unit of order 2 in $k[G]$. Note also that $u(1+x) = 1 + x + n + xn$ is a unit in $k[G]$ different from $1+x$ and $1+n$. Then we obtain that $E_x = \langle u, 1+x \rangle$ is isomorphic to E, and that $k[E_x] = k \oplus kn \oplus kx \oplus knx$ is contained in $k[G]$, where $nx = \nu_{E_x}$ is non-zero. We also have

$$\nu_{E_x} k[G] = k\nu_{E_x} + k(f-1)\nu_{E_x} = kv_x\nu_E + kv_x\nu_G,$$

which has dimension 2 as a vector space; hence the restriction $k[G]\downarrow_{E_x}$ is free. Similarly for y, the element

$$v_y = 1 + \frac{an_1 + cn_2 + s_1 n_e + s_2 n_{f^2}}{an_{f^2} + cn_e}(f-1)$$

is in $k[G]$. Then $\nu_{E_y} = (an_{f^2} + cn_e)v_y\nu_E$ is non-zero and $k[G]\downarrow_{E_y}$ is free.

(ii) Define $\phi_x : k[E_x] \longrightarrow k[E]$ by

$$\phi_x(u) = 1 + n_e(e-1) + n_{f^2}(f^2-1) + n_3\nu_E,$$
$$\phi_x(1+x) = 1 + a(e-1) + c(f^2-1) + r_3\nu_E.$$

Then $\phi_x(\nu_{E_x}) = (an_{f^2}+cn_e)\nu_E$ is non-zero. Hence ϕ_x is an isomorphism of group algebras. For similarly defined ϕ_y, we have the equivalence $\phi_x(x) \equiv \phi_y(y) \pmod{J_E^2}$. Thus we have the following equalities and isomorphisms

$$J_E/J_E^2 = k\overline{\phi_x(n)} \oplus k\overline{\phi_x(x)} = k\overline{\phi_y(n)} \oplus k\overline{\phi_y(y)} \cong J_{E_x}/J_{E_x}^2 \cong J_{E_y}/J_{E_y}^2.$$

□

Lemma 3.3. *Let x be in J.*

(i) *If x^2 is non-zero, then $\mathbb{J}^{(2)}$ is contained in xJ.*

(ii) *If x is not in J^2, then $\mathbb{J}^{(3)}$ is contained in xJ.*

Proof. Write x as in (6). Under the hypothesies of (i) or (ii) xJ contains $(e-1)J$ and $(f-1)J$.

(i) The hypothesis that x^2 is non-zero implies that b is non-zero; then $\mathbb{J}^{(2)}$ is a subset of $(f-1)J$.

(ii) By hypothesis, we have (a,b) different than $(0,0)$. If b is not zero, then the first part implies the result, because $\mathbb{J}^{(3)}$ is contained in $\mathbb{J}^{(2)}$. If a is not zero, then $\mathbb{J}^{(3)}$ is a subset of $(e-1)J$. □

The following corollary is a special case of Lemma 2.5 and is essential for proving Theorems 5.1 and 5.3. The cohomology algebras are as given in (2) and

$$H^*(C_2;k) = k[t], \quad H^*(C_4;k) = k[\tau] \otimes \Lambda(v).$$

We also write $k\langle g\rangle$ instead of $k[\langle g\rangle]$ for simplicity.

Corollary 3.4. (i) *The Heller shift $\Omega^n_{C_2}(k)$ and L_0 are isomorphic to k for all n while L_t is zero.*

(ii) *The dimensions of $\Omega^n_E(k)$ and L_{ξ^m} are $2n+1$ and $2m$, respectively, where ξ is in $H^1(E;k)$. Moreover, the following isomorphisms hold:*

$$\Omega^n_E(k)\downarrow_{\langle u_\beta\rangle}(k) \cong \Omega^n_{\langle u_\beta\rangle}(k) \oplus (k\langle u_\beta\rangle)^n \cong k \oplus (k\langle u_\beta\rangle)^n$$

and

$$(L_{\xi^m_\alpha})\downarrow_{\langle u_\beta\rangle} \cong \begin{cases} (k\langle u_\beta\rangle)^m, & \text{if } k\{\beta\} \neq k\{\alpha\} \text{ iff } \xi^m_\alpha\downarrow_{\langle u_\beta\rangle} \neq 0, \\ (k)^2 \oplus (k\langle u_\beta\rangle)^{m-1}, & \text{if } k\{\beta\} = k\{\alpha\} \text{ iff } \xi^m_\alpha\downarrow_{\langle u_\beta\rangle} = 0. \end{cases}$$

(iii) *When the group is C_4, L_τ is the zero module, and the Heller shifts are*

$$\Omega^n_{C_4}(k) \cong \begin{cases} k, & \text{if } n \text{ is even,} \\ \Omega^1_{C_4}(k) \cong J_{C_4}, & \text{if } n \text{ is odd.} \end{cases}$$

(iv) *When the group is G, the dimensions of the Heller shifts and $\mathbb{L}_\zeta$'s for ζ $H^n(G;k)$ are as follows:*

$$\dim(\Omega^n_G(k)) = \begin{cases} 4n+1, & \text{if } n \text{ is even,} \\ 4n+3, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\dim(\mathbb{L}_\zeta) = \dim(\Omega^n_G(k)) - 1.$$

When $\mathbb{L}_\zeta$ is restricted to E, it takes the form

$$\mathbb{L}_\zeta\downarrow_E \cong \begin{cases} L_{\zeta\downarrow_E} \oplus (k[E])^{n/2}, & \text{if } n \text{ is even,} \\ L_{\zeta\downarrow_E} \oplus (k[E])^{(n+1)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

For ζ_α in degree 2, the restrictions of $\mathbb{L}_{\zeta_\alpha}$ to the shifted cyclic subgroups $\langle \mathbb{C}_\gamma\rangle$ of $k[G]$ take the forms

$$\mathbb{L}_{\zeta^m_\alpha}\downarrow_{\langle\mathbb{C}_\gamma\rangle} \cong \begin{cases} (k\langle\mathbb{C}_\gamma\rangle)^{4m}, & \text{if } \zeta^m_\alpha\downarrow_{\langle\mathbb{C}_\gamma\rangle} \neq 0, \\ (k)^2 \oplus (k\langle\mathbb{C}_\gamma\rangle)^{4m-1}, & \text{if } \zeta^m_\alpha\downarrow_{\langle\mathbb{C}_\gamma\rangle} = 0, \end{cases} \quad \textit{where} \quad \langle\mathbb{C}_\gamma\rangle \cong C_2,$$

and

$$\mathbb{L}_{\zeta_\alpha^m}\!\downarrow_{\langle \mathbb{C}_\gamma\rangle} \cong \begin{cases} (k\langle \mathbb{C}_\gamma\rangle)^{2m}, & \text{if } \zeta_\alpha^m\!\downarrow_{\langle \mathbb{C}_\gamma\rangle} \neq 0, \\ k \oplus \Omega^1_{\langle \mathbb{C}_\gamma\rangle}(k) \oplus (k\langle \mathbb{C}_\gamma\rangle)^{2m-1}, & \text{if } \zeta_\alpha^m\!\downarrow_{\langle \mathbb{C}_\gamma\rangle} = 0, \end{cases} \quad \text{where} \quad \langle \mathbb{C}_\gamma\rangle \cong C_4.$$

Proof. (i) and (iii) follow from the fact that the minimal $k[C_{p^n}]$-free resolution of k is of the form

$$\cdots \longrightarrow k[C_{p^n}] \longrightarrow k[C_{p^n}] \longrightarrow k \longrightarrow 0$$

for any p and n.

(ii) follows from the classification of $k[E]$-modules (see [**Ca**]) and the definitions of $\Omega^n(k)$ and L_ζ.

(iv) follows from the fact that the minimal $k[G]$-free resolution of k is of the form

$$\cdots \longrightarrow (k[G])^3 \longrightarrow (k[G])^2 \longrightarrow (k[G])^1 \longrightarrow k \longrightarrow 0,$$

and the definitions of $\Omega^n(k)$ and L_ζ. □

4. Structure Theorems

In this section we prove our theorems for modules over $k[G]$ and $k[E]$ for our fixed group $G = C_2 \times C_4$ and its unique elementary subgroup $E = C_2 \times C_2$.

Theorems 4.1 and 4.3 contain generalizations of Dade's Lemma and Carlson's analoguous theorems for modules over elementary abelian p-groups. Theorems 4.5 and 4.7 guarantee the well-definedness of the set of multiplicities of a module over $k[G]$ and $k[E]$, respectively. In the proofs we use Corollary 2.2 or 2.3 to determine whether a module is free or not, and Lemmas 3.1, 3.3 for the properties of the elements of J. We use the notation of (6) for x in J.

Theorem 4.1. (i) *If $x \in J$, then $k[G]\!\downarrow_{\langle 1+x\rangle}$ is free if and only if $x \notin \mathbb{J}^{(2)}$.*

(ii) *If M is a $k[G]$-module, then M is free if and only if the restriction $M\!\downarrow_{\langle 1+x\rangle}$ is free for all x in $J \backslash \mathbb{J}^{(2)}$.*

Proof. Note that x is not in $\mathbb{J}^{(2)}$ if and only if (a, b, c) in (6) is not equal to $(0, 0, 0)$.

(i) By Corollary 2.2 (iii) $k[G]\!\downarrow_{\langle 1+x\rangle}$ is free if and only if $\dim(xk[G])$ is 4 or 6 depending on b is zero or not, respectively. Computing the rank of the matrix representing x shows that this dimension requirement is satisfied as $(a, b, c) \neq (0, 0, 0)$.

(ii) Suppose M is a free $k[G]$-module and $x \in J$. By (i) $k[G]\!\downarrow_{\langle 1+x\rangle}$ is free if and only if $x \notin \mathbb{J}^{(2)}$, hence $M\!\downarrow_{\langle 1+x\rangle}$ is free for all $x \notin \mathbb{J}^{(2)}$. Conversely assume that $M\downarrow_{\langle 1+x\rangle}$ is free for all $x \in J\backslash\mathbb{J}^{(2)}$. As usual, let E denote the maximal elementary abelian subgroup of G. Since $J_E \subset J$ and $J_E^2 \subset \mathbb{J}^{(2)}$, $M\downarrow_{\langle 1+z\rangle}$ is free for all $z \in J_E\backslash J_E^2$. Then by Dade's Lemma $M\downarrow_E$ is a free module. Hence M is a free $k[G]$-module by Chouinard's Theorem, Theorem 1.7, as E is the only maximal elementary abelian subgroup of G. □

Remark 4.2. The first part of the above theorem implies that the shifted cyclic subgroups of $k[G]$ are of the form $\langle 1+x\rangle$ for any $\bar{x}$ in $J/\mathbb{J}^{(2)}$. If the natural one-to-one correspondance between k^3 and $J/\mathbb{J}^{(2)}$ is used (see (1)), then shifted cyclic subgroups of G can also be defined as $\langle\mathbb{C}_\gamma\rangle$ for each $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ in k^3, where

$$\mathbb{C}_\gamma = 1 + \gamma_1(e-1) + \gamma_2(f-1) + \gamma_3(f^2-1). \tag{7}$$

Obviously, $\langle\mathbb{C}_\gamma\rangle$ is isomorphic to C_4 if and only if γ_2 is not zero; otherwise, it is isomorphic to C_2. When γ_2 is non-zero, then $\langle\mathbb{C}_\gamma^2\rangle$ is $\langle 1+\gamma_2^2(f^2-1)\rangle$. It is clear that the lines through the origin in k^3, parametrize the shifted cyclic subgroups of $k[G]$. However note that the points $(0,1,1,)$ and $(0,1,0)$ give the same group algebra, $k[\langle 1+f-1+f^2-1\rangle] = k\langle 1+f-1\rangle] = k\langle f\rangle]$. Thus we can use points of k^3 and $J/\mathbb{J}^{(2)}$ interchangebly to write the shifted cyclic subgroups of $k[G]$. The latter definition is consistent with Carlson's definition of shifted cyclic subgroups of $k[E]$, namely, $\langle u_\alpha\rangle$ for $\alpha = (\alpha_1, \alpha_2)$ in k^2, where $u_\alpha = 1+\alpha_1(e-1)+\alpha_2(f^2-1)$. The muliplicities set in this notation can be written as $\mathcal{N}_E(M) = \Big\{ \big[\alpha; \eta(u_\alpha - 1)\big] \mid \alpha \in k^2\backslash 0 \Big\}$.

Theorem 4.3. *Let M be a $k[G]$-module, x, y be in $J\backslash\mathbb{J}^{(2)}$, and $x \equiv y \pmod{\mathbb{J}^{(2)}}$.*

(i) *If x^2 is not zero, then $M\downarrow_{\langle 1+x\rangle} \equiv M\downarrow_{\langle 1+y\rangle}$.*

(ii) *$M\downarrow_{\langle 1+x\rangle}$ is free if and only if $M\downarrow_{\langle 1+y\rangle}$ is free.*

Proof. Since x is not in $\mathbb{J}^{(2)}$ (a,b,c) of (6) is not equal to $(0,0,0)$, hence $x \equiv y \pmod{J^2}$.

(i) Since x^2 is not zero, Lemma 3.3 implies that $\mathbb{J}^{(2)}$ is contained in xJ. Then $y = x(1-r)$ for some r in J as $x-y$ is in $\mathbb{J}^{(2)}$. This proves the claim because $1-r$ is a unit.

(ii) The assumption $x \equiv y \pmod{\mathbb{J}^{(2)}}$ implies that $x^2 = y^2$ (see Lemma 3.1 (ix)). In the case x^2 is non-zero, Theorem 4.3 (i) implies that $M\downarrow_{\langle 1+x\rangle} \equiv M\downarrow_{\langle 1+y\rangle}$. This, of course, implies that $M\downarrow_{\langle 1+x\rangle}$ is free if and only if $M\downarrow_{\langle 1+y\rangle}$ is free.

In the case x^2 is zero, without loss of generality we can write $x = a(e-1) + c(f^2-1)$ and $y = x+w$ where $w \in \mathbb{J}^{(2)}$. Note that $y^2 = 0$ and we can write $w = q \cdot r$ for some q, r in J with $q^2 = 0$, $r^4 = 0$.

Let z_M denote the map form M to M given by $z_M(m) = z \cdot m$ for all m in M, z in J.

Suppose $M\downarrow_{\langle 1+x\rangle}$ is free. Since $x^2 = 0$, we have $\ker(x_M) = xM$.

Claim: $M\downarrow_{\langle 1+y\rangle}$ is free. Let $N = \ker(y_M)/yM$, $y = qr$ with $q^2 = 0$, $r^4 = 0$ as written above. It remains to show that $N = 0$. Consider the map $r_N : N \longrightarrow N$, multiplication by r. Let $\bar{m}$ be in $\ker(r_N)$. Then we have m in $\ker(y_M)$ and rm in yM. Thus, $ym = (x+qr)m = 0$ and $rm = yn$ for some n in M. Since $q^2 = 0$, we obtain $xm = -qrm = qrm = q(x+qr)n = xqn$. Hence $m+qn$ is in $\ker(x_M)$. Thus $m + qn = xs$ for some s in M. Multiplying the last equation by r we obtain that $rm = rqn + rxs$. We had $rm = yn = xn + qrn$ above, hence $rxs = xrs = xn$. Thus $n - rs$ is in $\ker(x_M) = xM$. Hence we can write $n = rs + xt$ for some t in M. We have $m = qn + xs = q(rs+xt)+xs = qrs+qxt+xs+q^2rt = (x+qr)(s-qt) = y(s-qt)$. Thus $\bar{m} = \bar{0}$, showing that r_N is injective. On the other hand r is nilpotent, this forces N to be zero which proves the claim.

Similarly, $M\downarrow_{\langle 1+y\rangle}$ is free implies that $M\downarrow_{\langle 1+x\rangle}$ is free. $\square$

Remark 4.4. Suppose M is a $k[G]$-module. Then

$$V_E^r(M\downarrow_E) = \{0\} \cup \left\{ \overline{x} \in J_E/J_E^2 \;\middle|\; \eta_2(x) \neq \dim(M)/2 \right\}.$$

The restriction $M\downarrow_{\langle \mathbb{C}_\gamma\rangle}$ is free if and only if the restriction $M\downarrow_{\langle \mathbb{C}_\gamma^2\rangle}$ is free by Chouinard's Theorem, Theorem 1.7, which holds if and only if the restriction $M\downarrow_{\langle f^2\rangle}$ is free. Note that $\langle 1+f^2\rangle$ is the shifted cyclic subgroup of $k[E]$ corresponding to the point $(0,1)$.

Theorem 4.5. *Let M be a $k[G]$-module, x, y be in $J\backslash J^2$, and $x \equiv y \pmod{\mathbb{J}^{(3)}}$. Then*

$$M\downarrow_{\langle 1+x\rangle} \equiv M\downarrow_{\langle 1+y\rangle}\ .$$

Proof. By Lemma 3.3, we know that $\mathbb{J}^{(3)}$ is contained in xJ. Then $y = x(1-r)$ for some r in J as $x - y$ belongs to $\mathbb{J}^{(3)}$. This proves the claim as $1-r$ is a unit. $\square$

Remarks 4.6. **(1)** The hypothesis $x^2 \neq 0$ in Theorem 4.3 (i) can be replaced by any one of the **(a)** – **(e)** : **(a)** $xy = 0$ and $x^2 = 0$, **(b)** $x \equiv y \pmod{k\nu_E \oplus k\nu_G}$, **(c)** $M{\downarrow_{\langle e \rangle}}$ or $M{\downarrow_{\langle f \rangle}}$ or $M{\downarrow_{\langle f^2 \rangle}}$ is the trivial module, **(d)** $M{\downarrow_{\langle f \rangle}}$ has no free summands, **(e)** $(e-1)(f-1)M = 0$.

(2) Note that Theorem 4.5 makes it clear that the set of multiplicities is well-defined for a $k[G]$-module when we take $I_G = \mathbb{J}^{(3)}$ in equation (3). That is, for a $k[G]$-module M, its *set of multiplicities* is defined as

$$\mathcal{N}_G(M) = \Big\{ \big[x; \eta_1(x), \eta_2(x), \eta_3(x), \eta_4(x)\big] \bigm| \overline{x} \in J/\mathbb{J}^{(3)} \backslash 0 \Big\},$$

note that $\eta_3(x) = \eta_4(x) = 0$ for x in J with $x^2 = 0$. Further, we define a *filtration* of it by the subsets

$$\mathcal{N}_G^i(M) = \Big\{ \big[x; \eta_1(x), \eta_2(x), \eta_3(x), \eta_4(x)\big] \bigm| 0 \neq \overline{x} \in J/\mathbb{J}^{(3)},\ \eta_j(x) = 0 \text{ for } j \geq i \Big\}.$$

It is obvious that $\mathcal{N}_G^i(M)$ form a nested sequence

$$\mathcal{N}_G^1(M) \subset \mathcal{N}_G^2(M) \subset \mathcal{N}_G^3(M) \subset \mathcal{N}_G^4(M), \tag{8}$$

and $\mathcal{N}_G^i(M) \backslash \mathcal{N}_G^{i+1}(M)$ gives those x's for which $M{\downarrow_{\langle 1+x \rangle}}$ has only i-dimensional indecomposable summands. Recall that the Loewy length of a non-free indecomposable $k[G]$-module is at most 4, and the Loewy length of M is i if and only if $\eta_i(x)$ is non-zero for some x in $J \backslash J^2$, which in turn holds if and only if $\mathcal{N}_G(M) = \mathcal{N}_G^i(M)$. For a non-free $k[G]$-module M, we can write $\mathcal{N}_G(M) = \cup_i \mathcal{N}_G^i(M)$.

The analogue of Theorem 4.5 for the group E is given below which shows that $I_E = J^2$, see (3) for the definition of I_E.

Theorem 4.7. *Let M be a $k[E]$-module and let x, y be in $J \backslash J^2$ such that $x \equiv y \pmod{J^2}$. Then*

$$M{\downarrow_{\langle 1+x \rangle}} \equiv M{\downarrow_{\langle 1+y \rangle}} .$$

Proof. Recall that $E = \langle e, f^2 \rangle$. Without loss of generality, assume that M is non-free and indecomposable. Then the number of $k[E]$-free summands in M is zero; equivalently $(e-1)(f^2-1)M = J^2 M = 0$. Therefore $xm = ym$ for all m in M as $(x-y)m$ is in $J^2 M$; thus $M{\downarrow_{\langle 1+x \rangle}} \equiv M{\downarrow_{\langle 1+y \rangle}}$. □

5. Detection Theorems and Examples

This section is devoted to the applications of our definitions and theorems. These theorems are based on the observation that, when restricted to $\langle 1+x\rangle$, the indecomposable $k[E]$-module L_ζ have either two copies of the trivial module k or none, see (9) below. The proofs uses only Corollary 3.4, and Theorem4.7.

Theorem 5.1. *If M is a finitely generated isotypical $k[E]$-module, then $\mathcal{N}_E(M)$ determines M completely (up to isomorphism) except that isotypical modules of type $\Omega^n(k)$ and of type $\Omega^{-n}(k)$ can not be distinguished, where $\Omega^n(k)$ denotes the n-th Heller shift of the trivial module k.*

Proof. By the classification of $k[E]$-modules given in [**Ca**], a finitely generated indecomposable $k[E]$-module is isomorphic to one of the following: k, $k[E]$, $\Omega^n(k)$, $\Omega^{-n}(k)$, or $L_{\zeta_\alpha^n}$ for each $[\alpha]$ in $\mathbb{P}^1_k$, where $\alpha=(\alpha_1,\alpha_2)$ is in $k^2\backslash 0$, and n is a positive integer. The modules given have dimensions 1, 4, $2n{+}1$, $2n{+}1$, and $2n$, respectively, as vector spaces over k. Thus an isotypical $k[E]$-module M is isomorphic to m copies of one of the modules listed for some positive integer m. First we compute the set of multiplicities of each of the modules listed, then multiplying each multiplicity by m gives the set of multiplicities of M. We have the obvious isomorphisms

$$k{\downarrow_{\langle u_\beta\rangle}}\cong k \qquad \text{and} \qquad k[E]{\downarrow_{\langle u_\beta\rangle}}\cong (k\langle u_\beta\rangle)^2,$$

where u_β is a shifted cyclic subgroup of $k[E]$ and β is in $k^2\backslash\{0\}$. Therefore we have

$$\mathcal{N}_E(k)=\Big\{\,[\beta;1,0]\;\big|\;\beta\in k^2\backslash 0\Big\} \qquad \text{and} \qquad \mathcal{N}_E(k[E])=\Big\{\,[\beta;0,2]\;\big|\;\beta\in k^2\backslash 0\Big\}.$$

By Corollary 3.4 (ii), we know that the restrictions $\Omega^n(k){\downarrow_{\langle u_\beta\rangle}}$ and $\Omega^{-n}(k){\downarrow_{\langle u_\beta\rangle}}$ are both isomorphic to $k\oplus(k[\langle u_\beta\rangle])^n$. Hence the set of multiplicities for any two of them is $\{\,(\beta;1,n)\mid\beta\in k^2\backslash 0\,\}$. Therefore they cannot be distinguished by the set of multiplicities. By Corollary 3.4, we have the isomorphisms

$$L_{\xi_\alpha^n}{\downarrow_{\langle u_\alpha\rangle}}\cong (k)^2\oplus(k[C_2])^{(n-1)} \qquad \text{and} \qquad L_{\xi_\alpha^n}{\downarrow_{\langle u_\beta\rangle}}\cong (k[C_2])^n$$

for β not in $k\{\alpha\}$ which is the rank variety $V^r_E(L_{(\zeta_\alpha)^n})$ of $L_{(\zeta_\alpha)^n}$. Hence

$$\text{(9)} \qquad \mathcal{N}_E(L_{(\xi_\alpha)^n})=\Big\{\,[\alpha;2,n-1]\,\Big\}\cup\Big\{\,[(s,l);0,n]\;\big|\;(s,l)\in k^2\backslash\{0,\alpha\}\Big\}.$$

Therefore the possibilities for $\mathcal{N}_E(M)$ are

(i) $\{\,[\alpha;m,0]\;\big|\;\alpha\in k^2\backslash 0\,\}$,

(ii) $\{\, [\alpha; 0, 2m] \mid \alpha \in k^2\backslash 0 \,\}$,

(iii) $\{\, [\alpha; m, mn] \mid \alpha \in k^2\backslash 0 \,\}$,

(iv) $\{\, [\alpha; 2m, mn - m] \,\} \cup \{\, [(s,l); 0, mn] \mid (s,l) \in k^2\backslash\{0, \alpha\} \,\}$.

It is clear what the module should be in the first two cases. The third one implies M is either m copies of $\Omega_E^n(k)$ or $\Omega_E^{-n}(k)$. In the fourth case, η_1 can be zero or non-zero. In this case, $m = \eta_1/2$ and $n = (2\eta_2 + \eta_1)/\eta_1$ for the non-zero η_1 and the corresponding η_2. In each case m can be determined easily. Thus $\mathcal{N}_E(M)$ determines M (up to isomorphism) in each case. □

Examples 5.2. Let M be a $k[E]$-module. Item 2) below shows that if the hypothesis isotypical is removed from Theorem 5.1, then its conclusion is no longer true.

1) Let $M = (L_{t_1^3})^2$ and $M' = (L_{t_1^2})^3$ be $k[E]$-modules. Then by Corollary 3.4, M and M' are both of dimension 12 and they have the same rank variety, namely, the line $k\{(0,1)\}$. However, $[(0,1); 4, 4]$ is in $\mathcal{N}_E(M)$ and $[(0,c); 6, 3]$ is in $\mathcal{N}_E(M')$. Thus their set of multiplicities are not the same.

2) Let $M = L_{t_1^2+t_2^2} \oplus L_{t_1^5}$ and $M' = L_{t_1^4+t_2^4} \oplus L_{t_1^3}$ be two periodic $k[E]$-modules that are not isotypical. Then by Corollary 3.4, M and M' are both of dimension 14, they have the same rank variety, and they have the same set of multiplicities, namely,

$$\Big\{\, \big[(1,1); 2, 6\big], \big[(0,1); 2, 6\big] \,\Big\} \cup \Big\{\, \big[(s,l); 0, 7\big] \mid (s,l) \in k^2\backslash k\{(1,1)\} \cup k\{(0,1)\} \,\Big\}.$$

Theorem 5.3. *If M is a finitely generated isotypical $k[G]$-module of type $\mathbb{L}_{\zeta_\alpha^n}$ or induced from an isotypical $k[E]$-module of type $L_{\xi_\alpha^n}$, then M is completely determined (up to isomorphism) by its set of multiplicities $\mathcal{N}_E(M{\downarrow_E})$ when considered as a $k[E]$-module.*

Proof. Let $\alpha = (\alpha_1, \alpha_2)$ be in k^2 and $d = \dim(M)$. Then the shifted cyclic subgroup $< u_\alpha >$ is isomorphic to C_2. Suppose that M is m copies of $\mathbb{L}_{(\alpha_2 t_1^2 + \alpha_1 \tau)^n}$ for some m and α in $k^2\backslash 0$. By Corollary 3.4 (iv), we know $\dim(M) = m(4(2n)) = 8mn$, and for $\zeta = (\alpha_2 t_1^2 + \alpha_1 t_2^2)^n$, we have the isomorphisms

$$M{\downarrow_E} \cong (L_\zeta)^m \oplus (k[E])^{mn}, \;\; M{\downarrow_{\langle u_\alpha\rangle}} \cong (k)^{2m} \oplus (k[C_2])^{4mn-m}, \;\; M{\downarrow_{\langle u_\beta\rangle}} \cong (k[C_2])^{4mn}.$$

Hence

$$\mathcal{N}_E(M{\downarrow_E}) = \Big\{ \big[\alpha; 2m, 4mn - m\big] \Big\} \cup \Big\{ \big[\beta; 0, 4mn\big] \mid \beta \in k^2 \backslash \{\alpha, 0\} \Big\}.$$

Using the information on the non-free part, we obtain $m = \eta_1/2$, $n = 2\eta_2 + \eta_1/4\eta_1$, and $\dim(M) = 8mn$. Note that $|G/E| = 2$; then for

$$M \cong (L_{(\alpha_2 t_1 + \alpha_1 t_2)^n})^m {\uparrow_E^G}, \qquad \text{we have} \qquad M{\downarrow_{k[E]}} \cong (L_{(\alpha_2 t_1^2 + \alpha_1 t_2^2)^n})^{2m},$$

where m and α are in $k^2 \backslash 0$. By Corollary 3.4 (iv), we know $\dim(M{\downarrow_E}) = 4mn = \dim(M)$, and for $\zeta = (\alpha_2 t_1^2 + \alpha_1 t_2^2)^n$, we have the isomorphisms

$$M{\downarrow_E} \cong (L_\zeta)^{2m}, \quad M{\downarrow_{\langle u_\alpha \rangle}} \cong ((k)^2 \oplus (k[C_2])^{n-1})^{2m}, \quad M{\downarrow_{\langle u_\beta \rangle}} \cong (k[C_2])^{2mn}.$$

Thus

$$\mathcal{N}_E(M{\downarrow_E}) = \Big\{ \big[\alpha; 4m, 2mn - 2m\big] \Big\} \cup \Big\{ \big[\beta; 0, 2mn\big] \mid \beta \in k^2 \backslash \{\alpha, 0\} \Big\}.$$

Using the information on the non-free part, we obtain $m = \eta_1/4$, $n = 2\eta_2 + \eta_1/\eta_1$, and $d = 4mn$. In each case, n and m, and hence M are determined (up to isomorphism) by $\mathcal{N}_E(M{\downarrow_E})$. □

Examples 5.4. If we drop the hypothesis isotypical, then Theorem 5.3 fails, see the items 1) and 2) below.

1) Let $M = \mathbb{L}_{t_1^4 + \tau^2} \oplus \mathbb{L}_{t_1^4}$ and $M' = \mathbb{L}_{t_1^2 + \tau} \oplus \mathbb{L}_{t_1^6}$. By Corollary 3.4, we know that they are both of dimension 32, and we have the isomorphisms

$$M{\downarrow_E} \cong L_{t_1^4 + t_2^4} \oplus (k[E])^2 \oplus L_{t_1^4} \oplus (k[E])^2, \qquad M'{\downarrow_E} \cong L_{t_1^2 + t_2^2} \oplus k[E] \oplus L_{t_1^6} \oplus (k[E])^3.$$

Their rank variety as a module over $k[E]$ is a union $k\{(1,1)\} \cup k\{(0,1)\}$ of two lines. In addition, their sets of multiplicities are both equal to

$$\Big\{ \big[(0,c); 2, 15\big], \big[(c,c); 2, 15\big] \mid c \in k \backslash 0 \Big\}$$
$$\cup \Big\{ \big[(s,l); 0, 16\big] \mid (s,l) \in k^2 \backslash k\{(1,1)\} \cup k\{(0,1)\} \Big\}.$$

2) Let $M = L_{t_1^4 + t_2^4} {\uparrow_E^G} \oplus L_{t_1^3} {\uparrow_E^G}$ and $M' = L_{t_1^2 + t_2^2} {\uparrow_E^G} \oplus L_{t_1^5} {\uparrow_E^G}$. Since $|G/E| = 2$, we have the isomorphisms

$$M{\downarrow_E} \cong (L_{t_1^4 + t_2^4})^2 \oplus (L_{t_1^3})^2 \qquad \text{and} \qquad M'{\downarrow_E} \cong (L_{t_1^2 + t_2^2})^2 \oplus (L_{t_1^5})^2.$$

By Corollary 3.4 (ii), we know that M and M' are of dimension 28, and their rank variety as a module over $k[E]$ is the union of two lines, namely, $k\{(1,1)\} \cup k\{(0,1)\}$.

In addition, their sets of multiplicities are both equal to the set

$$\Big\{ \big[(0,c);4,12\big], ((c,c);4,12) \mid c \in k\backslash 0 \Big\} \cup \Big\{ \big[(s,l);0,14\big] \mid (s,l) \in k^2\backslash k\{(1,1)\} \cup k\{(0,1)\} \Big\}.$$

3) There are non-isomorphic indecomposable $k[G]$-modules M and M' such that their restriction to $k[E]$ are isomorphic, hence $\mathcal{N}_E(M) = \mathcal{N}_E(M')$. Let $M = \Omega^1(\mathbb{L}_{t_1^2})$ and $M' = \mathbb{L}_{t_1^2}$. We know that M' is of period 2, hence M is not isomorphic to M'. They both are isomorphic to $L_{t_1^2} \oplus k[E]$ when restricted to $k[E]$.

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Mathematics Department, Middle East Technical University, Ankara, 06531 , Turkey
E-mail address: `sozkap@metu.edu.tr`
URL: `http://arf.math.metu.edu.tr/~semra/`