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EQUIVALENT CONDITIONS FOR THE EXISTENCE OF NORMAL COMPLEMENTS

Semra Öztürk Kaptanoglu[1]
Mathematics Department
Middle East Technical University
Ankara 06531, Turkey
Fax: 90-312-210-1282
E-Mail: semra@math.metu.edu.tr

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Let H be a subgroup of a finite group G. A **normal complement to** H **in** G is a normal subgroup L of G such that $L \cap H = 1$ and $G = LH$. This problem has been studied by methods of group theory, character theory, and the restriction map in cohomology since the beginning of this century. In the literature, the usual hypotheses are either H has the character restriction property in G, or there is no G-fusion in H, or what we call the cohomological equivalence of H to G. This article is a result of the efforts to find out the interactions among these hypotheses. It was initiated by noticing the similarity of the character restriction property to the hypothesis of Theorem 1.4 in [Q]. It turns out that it is a natural consequence of the isomorphism given in [A] between the integral cohomology ring of G and the completion of the ring of $\mathbb{C}$-characters of G with respect to the augmentation ideal. We combine various results from the literature in Propositions 2.1 and 2.2. These include the cases for which the above three hypotheses on H are equivalent. Our contributions are the observation that similar equivalences exist for a nilpotent subgroup (Theorem 1.1), and versions of Proposition 2.1(e) (Theorem 1.2 and several corollaries).

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0. DEFINITIONS AND A BRIEF SURVEY

There are many results in the literature on the existence of a (normal) complement. We mention below some results due to Frobenious, Schur–Zassenhaus, Tate, Atiyah, Sah, Brauer, Friesen, Schmidt–Richen, Quillen, Jackowski, Hawkes–Humphreys, Isaacs, Henn–Lannes–Schwartz, Henn, and Mislin.

If H has a normal complement N in G, then any irreducible character of H can be lifted to an irreducible character of G whose kernel contains N. This necessary condition for the existence of a normal complement leads to the following definition. H is said to have the **character restriction** property G if every χ in the irreducible $\mathbb{C}$-characters $\mathrm{Irr}(H)$ of H is the restriction $\chi^*|_H$ of some $\chi^* \in \mathrm{Irr}(G)$ **[HH]**. Since characters are class functions, it follows from the character restriction property that any two G-conjugate elements of H are also H-conjugate that is $H \cap \mathrm{cl}_G(h) = \mathrm{cl}_H(h)$ for every $h \in H$. We say H has the **orbit** property in G if it satisfies this condition. If the above equality is true for the elements of H which has prime power order, we say H has the **restricted orbit** property in G. The name orbit is chosen because $\mathrm{cl}_G(h)$ and $\mathrm{cl}_H(h)$ are the orbits of h under the conjugation action of G and H respectively. What we call orbit property is referred to as H is c-closed in G in **[S]**. It is also referred to as no two elements of H are fused in H by G, or simply as there is no G-fusion in H. Among others, Frobenious **[F]** and Sah **[S]** assumed the orbit property as a part of their hypotheses to prove the existence of a normal complement to a Hall subgroup H in G. Recall also the Shur–Zassenhaus Theorem; if N is a normal Hall subgroup of G, then N has a complement in G and any two complements of N are conjugate in G. Obviously it is not necessary that $H \in \mathrm{Hall}(G)$ in order for H to have a normal complement in G. However, almost all the results had this assumption, usually together with the assumption G is a solvable group.

As usual let $\pi(G)$ denote the set of primes which divide the order of G. A subgroup H with the property that $\mathrm{res}_{G,H}$ in mod p cohomology induced by the inclusion $H \hookrightarrow G$ is an isomorphism for all $p \in \pi(H)$ is said to be **cohomologically equivalent** to G. Cohomological equivalence implies the restricted orbit property, see Theorems 2.1 and 2.2 in **[J]**. Note that S_4 is cohomologically equivalent to S_5, but S_5 has no normal subgroup of order 5 to complement S_4. Also S_4 does not have the character restriction property in S_5. On the other hand S_3 has a normal complement in S_4 without being cohomologically equivalent to S_4. It is not difficult to show that if $H \in \mathrm{Hall}(G)$ and H has a normal complement, then H is cohomologically equivalent to G, see Lemma 2.3. It follows from a theorem of Jackowski **[J]** that if H is cohomologically equivalent to G then H is a Hall subgroup. Brauer and Friesen proved that if G is a solvable group, $H \in \mathrm{Hall}(G)$ and H has the orbit property in G, then H has a normal complement in G. Jackowski **[J]** modified this result to make it work with the weaker assumption that H has the restricted orbit property. He also used the methods of **[A]** to show that if the restriction map $\mathrm{res}_{G,H}$ in mod p cohomoogy is an isomorphism in sufficiently high dimensions for all $p \in \pi(H)$, then H has the restricted orbit property in G. Thus he extended Quillen's result mentioned below to a Hall subgroup H of a solvable group G; see Theorems 2.1, 2.2, 3.3 and Corollary 2.4 in **[J]**.

Some examples of normal complement theorems that do not assume that H is Hall are due to Richen and Schmidt [**RS**], Sah [**S**]. Further generalizations of these are obtained by Hawkes and Humphreys [**HH**] and Isaacs [**I**] in the case G is a solvable group, H has the character restriction property in G and H is an $\mathcal{F}$-projector or an $\mathcal{F}$-normalizer when $\mathcal{F}$ is a saturated formation. For some formations $\mathcal{F}$, Hall subgroups or self-normalizing nilpotent subgroups appear as $\mathcal{F}$-projectors [**CH**]. See Theorem A in [**I**] for conditions in addition to H has the character restriction property in G which imply H has a normal complement in G and $H \in \mathrm{Hall}(G)$.

A normal complement to a $P \in \mathrm{Syl}(G)$ is said to be a **normal p-complement** for G. Quillen [**Q**] proved that G has a normal p-complement if the restriction homomorphism $\mathrm{res}_{G,P}$ in cohomology with $\mathbb{Z}_p$-coefficients induced by the inclusion $P \hookrightarrow G$ is an F-isomorphism, provided that p is odd. An F-isomorphism is a homomorphism with nilpotent kernel and cokernel. The hypothesis p is odd is due to a group theoretic result used in the proof which requires this hypothesis. Quillen's result extended Atiyah and Tate's cohomological results on the existence of a normal p-complement (Theorems 1.2 and 1.3 in [**Q**]). Later Henn, Lannes and Schwartz [**HLS**] proved that, for an odd p, G has a normal p-complement if and only if the map $\mathrm{Rep}(V, P)$ to $\mathrm{Rep}(V, G)$ induced by the inlusion of P in G is a bijection for every elementary abelian p-group V, where $\mathrm{Rep}(V, G)$ denotes the G-conjugacy classes of homomorphisms from V to G; see 4.2.3 in [**HLS**]. An extension of the above result of Quillen to compact Lie groups is obtained by Henn [**H**]. Compare Henn's results with those of Mislin [**M**] in which various equivalences involving $\mathrm{Rep}(V, G)$, $H^*(BG; \mathbb{Z}_p)$ etc. are given.

1. MAIN OBSERVATIONS

In the first theorem below we show that for a nilpotent subgroup of a group the concepts defined at the beginning coincide. Although the hypothesis that H is nilpotent is not necessary it cannot be replaced by H is solvable. As an example consider S_4 in S_5. S_4 is a non-nilpotent solvable Hall subgroup which is cohomologically equivalent to S_5 without having a normal complement in S_5. To generalize this theorem we consider subgroups H with 'nice' Sylow p-subgroups for every $p \in \pi(H)$.

Theorem 1.1. *For a nilpotent subgroup H of G, the following are equivalent:*

(i) H is cohomologically equivalent to G.

(ii) H has a normal complement and $H \in \mathrm{Hall}(G)$.

(iii) H has the orbit property and $H \in \mathrm{Hall}(G)$.

(iv) H has the character restriction property and $H \in \mathrm{Hall}(G)$.

(v) $\mathrm{Rep}(\rho) : \mathrm{Rep}(Q,H) \longrightarrow \mathrm{Rep}(Q,G)$ is a bijection for any p-group Q.

Proof. **(i) $\Longleftarrow$ (ii):** Follows from Lemma 2.3. **(i) $\Longrightarrow$ (ii):** Since H is nilpotent, $H \cong P_1 \times P_2 \times \cdots \times P_r$ and $\mathrm{res}_{H,P_i} : H^*(H;\mathbb{Z}_{p_i}) \longrightarrow H^*(P_i;\mathbb{Z}_{p_i})$ is an isomorphism for $P_i \in \mathrm{Syl}_{p_i}(H)$ and $p_i \in \pi(H)$. Then $\mathrm{res}_{G,P_i} = \mathrm{res}_{H,P_i} \circ \mathrm{res}_{G,H}$ is an isomorphism by the hypothesis that H is cohomologically equivalent to G. On the other hand $H \in \mathrm{Hall}(G)$ by Theorem 2.1 in [J], hence $P_i \in \mathrm{Syl}_{p_i}(G)$. Then P_i has a normal complement K_i in G by Theorem 1.2 and 1.3 in [Q]. These fit in an exact sequence

$$1 \longrightarrow G/(\textstyle\bigcap_{i=1}^r K_i) \longrightarrow G/K_1 \times \cdots \times G/K_r \cong P_1P_2\cdots P_r \cong H \longrightarrow 1.$$

Since $H \in \mathrm{Hall}(G)$, we have $\bigcap_{i=1}^r K_i \cap H = \{1\}$. Thus $\bigcap_{i=1}^r K_i$ is a normal complement to H. **(iii) $\Longleftrightarrow$ (ii):** By Theorem 3 in [S]. **(iv) $\Longleftrightarrow$ (ii):** By Theorem 2 in [S]. **(v) $\Longleftrightarrow$ (i):** By Theorem 3.13.

The equivalence of (i) and (ii) in Theorem 1.1 is a variation of Proposition 2.1 (d).

Theorem 1.2. *Let $H \leq G$ and assume that for every prime p, $P \in \mathrm{Syl}_p(H)$ is cohomologically equivalent to G. Then*

(i) H is cohomologically equivalent to G and H is nilpotent,

(ii) H has a normal complement and $H \in \mathrm{Hall}(G)$,

(iii) G has a normal p-complement for all $p \in \pi(H)$.

Proof. **(i)** Let $P \in \mathrm{Syl}_p(H)$. Then $\mathrm{res}_{H,P}$ is a monomorphism. Since P is cohomologically equivalent to G and $\mathrm{res}_{G,P} = \mathrm{res}_{H,P} \circ \mathrm{res}_{G,H}$ we obtain that $\mathrm{res}_{H,P}$ is an isomorphism, and hence $\mathrm{res}_{G,H}$ is an isomorphism. That is H is cohomologically equivalent to G. By Theorem 1.1 H has a normal p-complement for all $p \in \pi(H)$. This implies that H is nilpotent. **(ii)** By (i) H is nilpotent and cohomologically equivalent to G. Then the result follows from Theorem 1.1 (i) and (ii). **(iii)** Follows from Theorem 1.1 since P is nilpotent.

The p-rank of a group G is the maximal rank of of an elementary abelian p-subgroup of G.

Corollary 1.3. *Let G be a group and $H \in \mathrm{Hall}(G)$. Assume that $\pi(G) = \pi(H) \sqcup \mathcal{Q}$ and $p < q$ for every $p \in \pi(H)$, $q \in \mathcal{Q}$. Suppose, for $p \in \pi(H)$, a Sylow p-subgroup P of H is cyclic if $p = 2$, or p-rank of G is not greater than two if $p > 2$. Then H has a normal complement and H is cohomologically equivalent to G.*

Proof. Let p be the smallest prime divisor of H. By Theorem 7.6.1 in [**G**], G has a normal p-complement, say N. Then $H \cap N$ in N satisfies the hypotheses of the corollary. The result follows by induction on $|G|$ and Lemma 2.3.

Corollary 1.4. *Let $H \leq G$, $|H| = p_1 p_2 \cdots p_r$ with $p_1 < p_2 < \cdots < p_r$ distinct primes and $p_r < q$ for any $q \in \pi(G) \backslash \pi(H)$. Then H has a normal complement in G and H is cohomologically equivalent to G.*

Proof. Note that the hypothesis implies $H \in \mathrm{Hall}(G)$. Hence every $P_i \in \mathrm{Syl}_{p_i}(H)$ also satisfies $P_i \in \mathrm{Syl}_{p_i}(G)$. P_i is cyclic since $|P_i| = p_i$. Then the result follows from Corollary 1.3.

The following corollary was the motivating example for Corollary 1.3.

Corollary 1.5.

(i) If $S_3 \in \mathrm{Hall}(G)$, then S_3 has a normal complement in G and S_3 is cohomologically equivalent to G.

(ii) If S_3 is a cohomologically equivalent subgroup of G then S_3 has a normal complement in G.

(iii) If S_3 has a normal complement in G then the following are equivalent:

(1) S_3 is cohomologically equivalent to G.

(2) $S_3 \in \mathrm{Hall}(G)$.

Proof. (i) Note that $|S_3| = 2 \cdot 3$, and $2, 3$ are the smallest primes and the Sylow 2 and 3-subgroups of G are cyclic. The result follows from Corollary 1.3. **(ii)** Since S_3 is cohomologically equivalent to G, we have $S_3 \in \mathrm{Hall}(G)$ by Theorem 2.1 in [**J**]. Then by (i), S_3 has a normal complement in G. **(iii)** (1) $\Longrightarrow$ (2) by Theorem 2.1 in [**J**], (2) $\Longrightarrow$ (1) by Lemma 2.3.

2. SPECIAL CASES

Many results from various sources in the literature are brought together in the following propositions so that comparisons among similar results can be done easily.

Proposition 2.1. *Let $H \leq G$.*

(a) Suppose G and H satisfy one of the following conditions:

(i) $H \in \mathrm{Hall}(G)$ and H is solvable,

(ii) H is cohomologically equivalent to G and H is solvable,

(iii) G is a solvable group and H is a nilpotent subgroup with $N_G(H) = H$,

(iv) $H = N_G(P)$, where $P \in \mathrm{Syl}_p(G)$.

Then H has the character restriction property if and only if H has a normal complement.

(b) *Let π be a set of primes. Let H be a solvable π-subgroup, and suppose that H is maximal with this property. If H has the character restriction property in G, then H has a normal complement in G and H is a Hall π-subgroup of G.*

(c) *Let $H \in \mathrm{Hall}(G)$ and H be nilpotent. Then H has the orbit property if and only if H has a normal complement.*

(d) *Let G be a solvable group. Then H is cohomologically equivalent to G if and only if $H \in \mathrm{Hall}(G)$ and H has a normal complement.*

(e) *Let G be a solvable group with a cohomologically equivalent subgroup H whose Sylow subgroups are all cyclic. Then H has the restricted orbit property if and only if H has a normal complement.*

Proof. **(a)** (i) follows from Theorem 2 in [S]. (ii) follows from (i), because H is cohomologically equivalent to G implies that $H \in \mathrm{Hall}(G)$ by Lemma 2.3. (iii) For $\Longrightarrow$ see [**RS**]. $\Longleftarrow$ is true since existence of a normal complement always implies the character restriction property. (iv) For $\Longrightarrow$ see [I] Theorem B. For $\Longleftarrow$ see the proof of in (iii). **(b)** By Theorem A in [I]. **(c)** follows from Theorem 3.3 in [J]. **(d)** $\Longrightarrow$: By Theorem 2.1 in [J], $H \in \mathrm{Hall}(G)$. By Theorem 2.2 in [J], H has a restricted orbit property in G, then H has a normal complement by Theorem 3.3 in [J]. $\Longleftarrow$: By Lemma 2.3. **(e)** H is solvable by [G] 7.6.2. and $H \in \mathrm{Hall}(G)$ by Theorem 2.1 in [J]). $\Longleftarrow$: H has a normal complement implies that it has the character restriction property which implies the orbit property. $\Longrightarrow$: By Theorem 3.3 in [J].

Remark. Solvability of G in Proposition 2.1 (d) is necessary, S_4 in S_5 is an example. $S_4 \in \mathrm{Hall}(S_5)$, S_4 is cohomologically equivalent to S_5 but does not have a normal complement in S_5.

Proposition 2.2. *If either (i) H is a Sylow subgroup of a group G, or (ii) H is a Hall subgroup of a solvable group G, then the following are equivalent:*

(a) *H has a normal complement in G,*

(b) *H has the character restriction property in G,*

(c) *H has the (restricted) orbit property in G,*

(d) *H is cohomologically equivalent to G.*

Proof. **(i)**(a) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) by Theorem 1.1. (a) $\Longleftrightarrow$ (b) by Theorem 2 in [S]. **(ii)** (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) as mentioned in the proof of Proposition 2.1 (e). (c) $\Longrightarrow$ (a) by Theorem 3.3 in [J]. (d) $\Longleftrightarrow$ (a) by Proposition 2.1 (d). Also (a) $\Longrightarrow$ (d) by Lemma 2.3.

Lemma 2.3. *If $H \in \mathrm{Hall}(G)$ and has a normal complement, then H is cohomologically equivalent to G.*

Proof. Let K be a normal complement for H in G and $p \in \pi(H)$, then $p \nmid |K|$. Thus $H^q(K;\mathbb{Z}_p) = 0$ for $q > 0$. Using this in the Lyndon-Serre spectral sequence with $E_2^{m,n} = H^m(G/K; H^n(K;\mathbb{Z}_p))$ converging to $H^{m+n}(G;\mathbb{Z}_p)$ we obtain the isomorphism $H^*(G/K;\mathbb{Z}_p) \cong H^*(G;\mathbb{Z}_p)$. Note that $G/K \cong H$, so that $H^*(H;\mathbb{Z}_p) \cong H^*(G;\mathbb{Z}_p)$. Since $H \in \mathrm{Hall}(G)$, $\mathrm{tr\,ores}_{G,H} : H^*(G;\mathbb{Z}_p) \longrightarrow H^*(G;\mathbb{Z}_p)$ is an isomorphism.

Therefore $\mathrm{res}_{G,H}$ is injective. An injection between isomorphic $\mathbb{Z}_p$-spaces is necessarily onto hence isomorphism. Thus $\mathrm{res}_{G,H}$ is an isomorphism with $\mathbb{Z}_p$-coefficients for all $p \in \pi(H)$.

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