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BETTI NUMBERS OF FIXED POINT SETS AND MULTIPLICITIES OF INDECOMPOSABLE SUMMANDS

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Abstract

Let G be a finite group of even order, k be a field of characteristic 2, and M be a finitely generated kG-module. If M is realized by a compact G-Moore space X, then the Betti numbers of the fixed point set X^{C_n} and the multiplicities of indecomposable summands of M considered as a kC_n-module are related via a localization theorem in equivariant cohomology, where C_n is a cyclic subgroup of G of order n. Explicit formulas are given for $n = 2$ and $n = 4$.

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0. Introduction

Throughout the paper G denotes a finite group of order divisible by a prime p, A a subgroup of G, k a field of characteristic p, J the Jacobson radical of the group algebra kG, M a finitely generated kG-module, X a G-space, and X^A the fixed point set of A in X. Topological spaces with a G-action give rise to G-modules; for example, the cohomology group $H^i(X;k)$ with k-coefficients is a finitely generated kG-module for $i \geq 0$ provided that X is a compact G-space. Equivariant cohomology $H^*_G(X;k)$ of X is defined as the cohomology $H^*(X_G;k)$ of the Borel construction $X_G = (X \times EG)/G$ of X. When X is a point, we simply write H^*_G for $H^*_G(X;k)$ which is the same as $H^*(G;k)$. The constant map from X to a one-point space induces an H^*_G-module structure on $H^*_G(X;k)$. When G is an elementary abelian p-group and X is finite-dimensional, the inclusion map $j : (X^G, x_0) \hookrightarrow (X, x_0)$ induces an isomorphism in the localized equivariant cohomology of H^*_G-modules ([Qu]). A simply connected

G-space X is called a *G-Moore space* if $H^i(X, x_0; k) = 0$ for all i except for some fixed $n \geq 2$. A kG-module M is called *realizable* (in dimension n) if there exists a G-Moore space X whose cohomology in dimension n is M for some $n \geq 2$.

Suppose that M is a kG-module realized by X in dimension n. Then $M\downarrow_{kA}$, M considered as a kA-module, is also realized by X, and $H^*(A; M)$ is isomorphic to the equivariant cohomology ring $H_A^{*+n}(X, x_0; k)$. Combining this with the above isomorphism obtained by localization, of course for a 'nice' A or a 'nice' A-action (for example A acting *semi-freely* on X, that is, the isotropy subgroups being either A or $\{1\}$), we observe that the multiplicities of the indecomposable modules appearing in the decomposition of $M\downarrow_{kA}$ have a geometric interpretation in terms of the total Betti number β of the fixed point set X^A.

THEOREM. *Let G be a finite group of order divisible by 2, and C be a cyclic subgroup of G. Suppose that M is realized in dimension n by a compact space X. Then the following can be stated for the total Betti number β and the Euler characteristic χ of the fixed point set X^C of C:*

(a) *If $C \cong \mathbb{Z}_2$, then $\beta(X^C) = \eta_1 + 1$, where $M\downarrow_{kC} \cong (k)^{\eta_1} \oplus (kC)^{\eta_2}$.*

(b) *If $C \cong \mathbb{Z}_4$ and C acts semi-freely on X, then*

  (i) *$\beta^{\text{odd}}(X^C)$ is η_1 or η_3 if n is odd or if n is even, respectively, and $\beta(X^C) = \eta_1 + \eta_3 + 1$,*

  (ii) *$\chi(X^C) = (-1)^n(\eta_1 - \eta_3) + 1$,*

where $M\downarrow_{kC} \cong (k)^{\eta_1} \oplus (J^2)^{\eta_2} \oplus (J)^{\eta_3} \oplus (kC)^{\eta_4}$.

The restriction on the order of the cyclic subgroup C to be 2 or 4 in the theorem is due to the fact that for large orders that are powers of a prime $p \geq 2$, one could still obtain an isomorphism $H_C^*(X^C, x_0; k)[1/t] \cong H^*(C; M\downarrow_{kC})[1/t]$. However, interpreting the right hand side of the isomorphism to obtain a similar formula is not possible without such restrictions.

A corollary of the theorem is given in the discussion section.

1. Proof of Theorem

DEFINITION. Let S be a multiplicative subset of the polynomial part of H_G^* containing $1 \in H_G^*$, and G_x be the isotropy subgroup consisting of all $g \in G$ with $gx = x$. Define $X^S = \{x \in X : \ker\{\text{res} : H_G^* \to H_{G_x}^*\} \cap S = \emptyset\}$ following [Hs].

In some cases X^S turns out to be the same as the fixed point set X^A for some $A \leq G$; see [DW].

PROPOSITION 1. *Let G be a compact Lie group, X be a compact G-space, and $Y \subseteq X$ be a G-invariant subspace. Let $S \subset H_G^*$ be a multiplicative system. Then the localized homomorphism*

$$\rho^{-1} = S^{-1}i^* : S^{-1}H_G^*(X, Y) \to S^{-1}H_G^*(X^S, Y^S)$$

is an isomorphism, where i^* is the induced map in G-equivariant cohomology by the inclusion map $i : (X^S, Y^S) \hookrightarrow (X, Y)$.

PROOF. Recall that localization is an exact functor, and $\rho = S^{-1}i_G^* : S^{-1}H_G^*(X) \to S^{-1}H_G^*(X^S)$ is an isomorphism, where i_G^* is the map induced by the inclusion $i : X^S \hookrightarrow X$ in G-equivariant cohomology. Apply [Hs, Theorem III.1] to the long exact sequence of a pair in cohomology. The result then follows by the Five-Lemma. □

PROPOSITION 2. *Let M be a kG-module realized by X in dimension n. Then $H_G^{*+n}(X, x_0; k) \cong H^*(G; M)$.*

PROOF. Consider the Serre spectral sequence for the fibration $(X, x_0)_G = ((X, x_0) \times EG)/G \to EG/G = BG$ with fiber (X, x_0). Here EG is a contractible space on which G acts (fixed-point) freely. The spectral sequence has $E_2^{p,q}$-term equal to $H^p(G; H^q(X, x_0; k))$. For $q \neq n$, we have $H^q(X, x_0; k) = 0$; then $E_2^{p,q} = 0$ for $q \neq n$. Hence the sequence contains only one line and collapses. It follows that $E_2^{p,n} = H^p(G; H^n(X, x_0; k)) \cong H^p(G; M)$. Therefore $H_G^{*+n}(X, x_0) := H^{*+n}((X, x_0)_G; k) \cong H^*(G; M)$. □

PROOF OF THEOREM. Without loss of generality we may assume that X^G is non-empty; so let x_0 be in $X^G \subseteq X^K$ for $K \leq G$. Also X is a K-Moore space with $H^*(X; x_0) \cong M\downarrow_{kK}$ for $K \leq G$. Hence $H_K^{*+n}(X, x_0) \cong H^*(K; M\downarrow_{kK})$ by Proposition 2.

(a) Let $H_C^* = H^*(C; k) = k[t]$. By Proposition 1, localization with respect to $S = \{t^i : i \geq 0\}$ gives $H_C^*(X, x_0)[1/t] \cong H_C^*(X^C, x_0)[1/t]$. Since $\mathrm{res}_{C,\{1\}}(t) = 0$, we have $k[1/t] = 0$. Hence η_2 disappears after localization and we obtain $\dim_k H^*(X^C, x_0; k) = \beta(X^C) - 1 = \eta_1$, that is, $\beta(X^C) = \eta_1 + 1$.

(b) It is sufficient to prove only (i) since $\chi(X^C) = \beta^{\mathrm{even}}(X^C) - \beta^{\mathrm{odd}}(X^C)$. Let $C_2 \leq C$ and $C_2 \cong \mathbb{Z}_2$; let also $H_C^* = k[\tau'] \otimes \wedge(v')$ and $H_{C_2}^* = k[t]$. Thus $\mathrm{res}_{C,C_2}(\tau') = t^2$. We have $H^*(C; M\downarrow_{kC}) \cong (H_C^*)^{\eta_1} \oplus (H_{C_2}^*)^{\eta_2} \oplus (H^*(C; J))^{\eta_3} \oplus (k)^{\eta_4}$ since $J^2 \cong k[C/C_2] \cong k\uparrow_{kC_2}^{kC}$ and Shapiro's Lemma implies $H_{C_2}^* \cong H^*(C; J^2)$. Applying Proposition 1 with the multiplicative set $S = \{(\tau')^i : i \geq 0\}$ gives $H_C^*(X^{C_2}, x_0)[1/\tau'] \cong H_C^*(X, x_0)[1/\tau']$. The term with η_4 disappears after localization as in part (a). Hence

$$H_C^*(X^{C_2}, x_0)\left[\frac{1}{\tau'}\right] \cong \left(H_C^*\left[\frac{1}{\tau'}\right]\right)^{\eta_1} \oplus \left(H_{C_2}^*\left[\frac{1}{t^2}\right]\right)^{\eta_2} \oplus \left(H^*(C; J)\left[\frac{1}{\tau'}\right]\right)^{\eta_3}.$$

The hypothesis that C acts semi-freely on X implies $X^C = X^{C_2}$. Write $\hat{H}_C^* = H_C^*[1/\tau']$ and $\hat{H}_{C_2}^*[1/t]$. Then

$$(*) \qquad (\hat{H}_C^{*-n})^{\eta_1} \oplus (\hat{H}_{C_2}^{*-n})^{\eta_2} \oplus \left(H^{*-n}(C;J)\left[\frac{1}{\tau'}\right]\right)^{\eta_3} \cong H^*(X^C, x_0) \otimes \hat{H}_C^*.$$

Since $H^i(C;J) \cong H^{i-1}(C;k) = H_C^{i-1}$ for $i \geq 2$ and $H_C^{\text{odd}} = v'H_C^{\text{even}}$, we get $H^i(C;J)\cdot v' = 0$ for i even. Also $H_{C_2}^* \cdot v' = H_{C_2}^* \cdot \text{res}_{C,C_2}(v') = H_{C_2}^* \cdot 0 = 0$. Then $(*)$ becomes

$$(\hat{H}_C^{l-n}\cdot v')^{\eta_1} \oplus (\hat{H}_C^{l-n-1}\cdot v')^{\eta_3} \cong \sum_{i\geq 0, i \text{ even}}^{l} H^{l-i}(X^C, x_0) \otimes \hat{H}_C^i \cdot v'.$$

In particular,

$$\sum_{j\geq 0, j \text{ even}}^{l} H^{l-j}(X^C, x_0) \otimes \hat{H}_C^j \cdot v' \cong \begin{cases} (k)^{\eta_3}, & \text{if } l-n \text{ is odd;} \\ (k)^{\eta_1}, & \text{if } l-n \text{ is even.} \end{cases}$$

Choose an integer $l > \operatorname{Hom}\dim(X^C)$. For l even and l odd, we respectively obtain that

$$\beta^{\text{even}}(X^C) = \begin{cases} \eta_3 + 1, & \text{if } n \text{ is odd;} \\ \eta_1 + 1, & \text{if } n \text{ is even;} \end{cases}$$

and

$$\beta^{\text{odd}}(X^C) = \begin{cases} \eta_1, & \text{if } n \text{ is odd;} \\ \eta_3, & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof of the theorem. □

2. Discussion

The theorem of the paper is more meaningful when put in the context of the realization problem referred to in the literature as Steenrod's Problem, and/or in the classification problem of some category of kG-modules when G contains cyclic subgroups of order 2 and/or 4. (See the corollary below.) When G is a cyclic p-group of order p^n, all indecomposable kG-modules (up to isomorphism) are given by the powers of the Jacobson radical, namely, the ideals J^{p^n-i} of k-dimension i for $i = 1, \dots, p^n$. However, when G contains $\mathbb{Z}_p \times \mathbb{Z}_p$ there are infinitely many indecomposable kG-modules ([Hi]). Due to the lack of a classification for kG-modules when $G \supseteq \mathbb{Z}_p \times \mathbb{Z}_p$ except for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, considering the restrictions $M\downarrow_{kA}$ for various subgroups A in G to obtain information on M is a fundamental technique n modular representation theory. For example, the complexity of a kG-module, n particular, the cohomology $H^*(G;k)$ of the trivial kG-module k is 'detected' on maximal elementary abelian subgroups of G by theorems due to Quillen [Qu], Chouinard [Ch], and Alperin-Evens [AlEv]. See [Ka] for another detection theorem when $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. Furthermore, it is possible to obtain information on a kE-module M by considering $M\downarrow_{k\langle 1+x\rangle}$ for $x \in J\backslash J^2$ of kE, where E is an elementary abelian p-group [Ca]. See also [W].

Some partial results on Steenrod's Problem are as follows. All $k\mathbb{Z}_{p^m}$-modules are realizable (see [Ar]) and all realizable $k\mathbb{Z}_2 \times \mathbb{Z}_2$-modules are described in [BeHa]. When $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a normal Sylow subgroup of a finite group G, a kG-module M is realizable if and only if $M\downarrow_{k\mathbb{Z}_2\times\mathbb{Z}_2}$ is realizable ([Cn]). When G contains $\mathbb{Z}_p \times \mathbb{Z}_p$, there are kG-modules that are not realizable (see [Vo, Cs, As1, As2, BeHa]). Compare our theorem with [As3, Theorem 2.2], which states that the total Betti number $\beta(X^A)$ of a 'nice' Moore space X realizing a kE-module M is equal to the rank$(\mathcal{F}_A)$, where $\mathcal{F}_A$ is the characteristic sheaf of X and A is a subgroup of the elementary abelian p-group E.

The simplest group for which one can attack the classification problem or the realization problem for kG-modules is $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ due to the fact that it contains $\mathbb{Z}_2 \times \mathbb{Z}_2$ as its unique maximal elementary abelian subgroup and that the classification of $k\mathbb{Z}_2 \times \mathbb{Z}_2$-modules is known. As mentioned above, a 'detection' theorem supporting the first expectation is given in [Ka]. For the latter, we can only give a necessary condition for a $k\mathbb{Z}_2 \times \mathbb{Z}_4$-module M to be realizable by combining [Cs, Proposition II] and [Se, Proposition 1]: Let M be a $k\mathbb{Z}_2 \times \mathbb{Z}_4$-module. If $M\downarrow_{k\mathbb{Z}_2\times\mathbb{Z}_2}$ is realizable by X, then the rank variety $V^r_{\mathbb{Z}_2\times\mathbb{Z}_2}(M\downarrow_{k\mathbb{Z}_2\times\mathbb{Z}_2})$ (see [Ca]) is a union of $\mathbb{F}_2$-rational lines in k^2. Therefore for a realizable $k\mathbb{Z}_2 \times \mathbb{Z}_4$-module M, we obtain that $M\downarrow_{kS}$ is free for every shifted cyclic subgroup S of $k\mathbb{Z}_2 \times \mathbb{Z}_4$ except possibly for cyclic subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$. This can be used to construct non-realizable modules. Consider the induced $k\mathbb{Z}_2 \times \mathbb{Z}_4$-module $M_\alpha = k \otimes_{k\langle u_\alpha\rangle} k\mathbb{Z}_2 \times \mathbb{Z}_4$ for $\alpha \in k^2$. It can be seen easily by Mackey's formula that $V^r_{\mathbb{Z}_2\times\mathbb{Z}_2}(M_\alpha\downarrow_{k\mathbb{Z}_2\times\mathbb{Z}_2}) = k\{\alpha\}$ for $\alpha \in k^2$. Therefore, M_α is not realizable if α is not an $\mathbb{F}_2$-rational point.

The Theorem of this paper and the necessary condition mentioned above gives the following.

COROLLARY. *Let* $G = \langle e, f \,:\, e^2 = f^4 = efef^3 = 1\rangle \supset E = \langle e, f^2\rangle$. *If* M *is a non-free indecomposable* kG*-module realized by* X, *then* M *is a periodic* kG*-module, and* $M\downarrow_{k\langle 1+\alpha_1(e-1)+\alpha_2(f^2-1)\rangle}$ *is a free* $k\langle 1+\alpha_1(e-1)+\alpha_2(f^2-1)\rangle$*-module for* $(\alpha_1, \alpha_2) \in k^2$ *except possibly for* $(\alpha_1, \alpha_2) \in k\{(1,0)\} \cup k\{(0,1)\} \cup k\{(1,1)\}$. *Moreover, if* $M\downarrow_{k\langle g\rangle}$ *is a free* $k\langle g\rangle$*-module for* $g \in \{e, f^2, ef^2\}$, *then* $X^{\langle g\rangle}$ *is homotopic to a point.*

PROOF. The necessary condition given above for the realizability of a module M implies that $V = V^r_E(M\downarrow_{kE}) \subseteq k\{(1,0)\} \cup k\{(0,1)\} \cup k\{(1,1)\}$. This forces M to be periodic as it is indecomposable and non-free. In addition, since $k\langle 1+\alpha_1(e-1)+\alpha_2(f^2-1)\rangle$ for $\alpha \in \{(1,0)\} \cup k\{(0,1)\} \cup k\{(1,1)\}$ corresponds to $k\langle g\rangle$ for some $g \in \{e, f^2, ef^2\}$, it follows that $M\downarrow_{\langle g\rangle}$ is not free for at most one $g \in \{e, f^2, ef^2\}$. Suppose $M\downarrow_{\langle g\rangle}$ is a free $k\langle g\rangle$-module with $g \in \{e, f^2, ef^2\}$. Then it has no trivial summands, that is, $\eta_1 = 0$. Hence $\beta(X^{\langle g\rangle}) = 1$ by the theorem, and this implies that $X^{\langle g\rangle}$ is homotopic to a point. □

CONJECTURE. *If M is a finitely generated periodic $k\mathbb{Z}_2 \times \mathbb{Z}_4$-module, then M is realizable.*

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