COMMUTING NILPOTENT OPERATORS AND MAXIMAL RANK

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Abstract. Let X, $\widetilde{X}$ be commuting nilpotent matrices over k with nilpotency p^t, where k is an algebraically closed field of positive characteristic p. We show that if $X-\widetilde{X}$ is a certain linear combination of products of commuting nilpotent matrices, then X is of maximal rank if and only if $\widetilde{X}$ is of maximal rank.

1. Introduction

Let X be a nilpotent $d \times d$ matrix over a field of characteristic $p > 0$ having nilpotency p^t, i.e. $X^{p^t} = 0$, $X^{p^t-1} \neq 0$. The *Jordan type*, J-type(X), of X is defined as

$$J\text{-type}(X) = \eta_{p^t}[p^t] \oplus \eta_{p^t-1}[p^t-1] \oplus \cdots \oplus \eta_1[1],$$

where $[i]$ denotes the Jordan block of size $i \times i$ corresponding to the eigenvalue 0 and η_i denotes the multiplicity of $[i]$ in the Jordan canonical form of X.

A nilpotent matrix with nilpotency p^t is said to be of *maximal rank (or of maximal Jordan type)* if its Jordan canonical form consists only of Jordan blocks of size p^t, that is, its Jordan type is $\eta_{p^t}[p^t] \oplus 0[p^t-1] \oplus \cdots \oplus 0[1]$. Thus the largest possible rank that a $d \times d$ matrix of nilpotency p^t can attain is $d\frac{p^t-1}{p^t}$.

Main Theorem. *Let $X, X_1, \ldots, X_s$ be $d \times d$ pairwise commuting nilpotent matrices over a field of characteristic p with nilpotencies $p^n, p^{n_1}, \ldots, p^{n_s}$ respectively, where $n \geq n_1 \geq n_2 \geq \cdots \geq n_s$. Let $f \in k[t_1, \ldots, t_s]$ be a polynomial having no constant or linear terms and $\widetilde{X} = X + f(X_1, \ldots, X_s)$. Then the matrix X is of maximal rank if and only if the matrix $\widetilde{X}$ is of maximal rank.*

An $d \times d$ nilpotent matrix with nilpotency p^t represents the action of $g-1$ on a k-vector space of dimension d for the group $\langle g \rangle \cong C_{p^t}$. In other words, it represents a $k[C_{p^t}]$-module of k-dimension d. Note also that a $k[C_{p^t}]$-module is free if and only if the matrix representing the action of $g-1$ is of maximal rank. Therefore Main Theorem is a restatement of Theorem 3.2.

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Let $k[G]$ denote the group algebra of the group G, and C_n denote the cyclic group of order n. There has been studies of $k[G]$-modules in terms of the Jordan type of the restriction of the module to subalgebras of $k[G]$ that are of the form $k[C_p] \cong k[x]/(x^p)$ even though it it is not referred as Jordan type; when the group G is an elementary abelian p-group, i.e. $G \cong (C_p)^{\times n}$, in [**Da**] Dade gave a criterion to determine the freeness of a $k[G]$-module in terms of certain $k[C_p]$'s, in [**Ca**] Carlson introduced the rank variety for a $k[G]$-module when G is an elementary abelian p-group. Generalizations of Carlson's work to infinitesimal group schemes, and finite group schemes can be found in [**SFB**], [**FP**] respectively. In [**SFP**] generic and maximal Jordan types for modules are studied; this is followed by [**CFP**] where modules of constant Jordan type are introduced; exact category of modules of constant Jordan type are studied in [**CF**]. Studying modules by way of Jordan types is an active research area. Recently this type of study has been generalized to include the restrictions to subalgebras of $k[G]$ that are of the form $k[C_{p^t}]$ for $t \geq 1$, which in turn led to the definition of modules of constant p-power Jordan type when G is an abelian p-group in [**Ka**].

2. Preliminaries

We first give two lemmas from the literature; the first one is for determining the freeness of a $k[G]$-module, the second one is a result on binomial coefficients mod-p.

Let H be a finite group. The element $\nu_H := \sum_{h \in H} h$ of $k[H]$ is referred as the *norm* element of the group algebra $k[H]$. Note that if $H = \langle g \rangle$, where g is of order p^m, then $\nu_H = (g-1)^{p^m-1}$ and $\nu_{\langle g^{p^{(m-1)}} \rangle} = (g^{p^{(m-1)}} - 1)^{p-1} = (g-1)^{p^m - p^{(m-1)}}$.

Lemma 2.1. *Let P be a finite p-group, and M be a finitely generated $k[P]$-module. Then $\dim_k(\nu_P M) \leq \frac{1}{|P|} \dim_k(M)$. Moreover, the following are equivalent.*

(i) $\dim_k(\nu_P M) = \frac{\dim_k(M)}{|P|}$.

(ii) *M is a free $k[P]$-module.*

In particular, if $P = \langle g \rangle$ is of order p^m, then the following are equivalent.

(iii) $\dim_k((g-1)^{p^m-1}M) = \frac{\dim_k(M)}{p^m}$.

(iv) $\dim_k((g-1)M) = (p^m - 1)\frac{\dim_k(M)}{p^m}$.

(v) $\ker(g-1 \text{ on } M) = (g-1)^{p^m-1}M$.

(vi) $\ker((g-1)^{p^m-1} \text{ on } M) = (g-1)M$.

Proof. The equivalences can be derived from the Jordan form of the matrix representing the action of $g-1$ on M and the equivalent definitions of freeness. ∎

Lemma 2.2. *Let a, b be elements of a commutative k-algebra with $char(k) = p > 0$, and m be a positive integer. Then*

$$(a+b)^{p^m-1} = a^{p^m-1} - a^{p^m-2}b + \cdots - ab^{p^m-2} + b^{p^m-1}.$$

Proof. It is known that the binomial coefficients satisfy the following congruence; $\binom{p^n-1}{j} \equiv (-1)^j \mod (p)$. □

3. Proof of the Main Theorem

First we prove a lemma using the lemmas given in the Preliminaries. It is the key result in the proof of the Main Theorem. It is a generalization of Proposition 2.2 of [**FP**] which is a generalization of Lemma 6.4 of [**Ca**], and also of Proposition 3.1 of [**Fr**].

Lemma 3.1. *Let x, y, z be pairwise commuting nilpotent operators on $M = k^d$ where k is a field of characteristic p. Suppose that the nilpotencies of x, y, z are p^{m_0}, p^{m_1}, l, respectively, where $m_0 \geq m_1$ and $l \geq 1$. Then M is free as a $k[C_{p^{m_0}}]$-module where the action of $g-1$ on M is given by x if and only if M is free as a $k[C_{p^{m_0}}]$-module where the action of $g-1$ on M is given by $x+yz$ for a generator g of $C_{p^{m_0}}$.*

Proof. Note that $m_0 \geq m_1$ and k is a field of characteristic p, $(x+yz)^{p^{m_0}} = 0$.

$\Longrightarrow$: Suppose that M is free as a $k[C_{p^{m_0}}]$-module where the action of $g-1$ on M is given by x. By Lemma 2.1 (v), this means that $\ker(x)$ on M is equal to $x^{p^{m_0}-1}M$ as well as $\ker(x^{p^{m_0}-1})$ on M is equal to xM. Let $N = \ker(x+yz)/(x+yz)^{p^{m_0}-1}M$. To show M is free as a $k[C_{p^{m_0}}]$-module where the action of $g-1$ on M is given by $x+yz$, we need to show that $N = 0$. Consider the action of z on N.

Claim: The action of z on N is injective. Suppose $m \in \ker(x+yz)$ and $zm \in (x+yz)^{p^{m_0}-1}M$. It suffices to show that $m \in (x+yz)^{p^{m_0}-1}M$. By the hypothesis $xm = -yzm$ and $zm = (x+yz)^{p^{m_0}-1}n$ for some $n \in M$. Hence by Lemma 2.2 we have

$$xm = -y(x^{p^{m_0}-1} - x^{p^{m_0}-2}yz + \cdots - x(yz)^{p^{m_0}-2} + (yz)^{p^{m_0}-1})n.$$

Since $y^{p^{m_0}} = 0$, $xm = -y(x^{p^{m_0}-1} - x^{p^{m_0}-2}yz + \cdots - x(yz)^{p^{m_0}-2})n$. Factoring x gives $xm = -yx(x^{p^{m_0}-2} - x^{p^{m_0}-3}yz + \cdots - (yz)^{p^{m_0}-2})n$; thus

$$m + y(x^{p^{m_0}-2} - x^{p^{m_0}-3}yz + \cdots - (yz)^{p^{m_0}-2})n \in \ker(x).$$

Since $\ker(x) = x^{p^{m_0}-1}M$, $m = x^{p^{m_0}-1}s - y(x^{p^{m_0}-2} - x^{p^{m_0}-3}yz + \cdots - (yz)^{p^{m_0}-2})n$ for some $s \in M$. Multiplying m by z in the last equation and using the fact that $zm = (x+yz)^{p^{m_0}-1}n$ for some $n \in M$, we obtain that $n - zs \in \ker(x^{p^{m_0}-1})$. Thus

$n = xt + zs$ for some $t \in M$ as $\ker(x^{p^{m_0}-1}) = xM$ by Lemma 2.1 (vi). It follows that $m = (x + yz)^{p^{m_0}-1}(s - yt)$, hence the claim is proved.

Since z is nilpotent, and injective on N, we conclude that $N = 0$.

$\Longleftarrow$ Let $x' = x + yz$, $y' = -y$, $z' = z$. Arguing as in the proof of the other direction with with x', y', z' instead of x, y, z, respectively, we get the desired result. Namely, if M is free as a $k[C_{p^{m_0}}]$-module where the action of $g - 1$ on M is given by $x' = x + yz$ then M is free as a $k[C_{p^{m_0}}]$-module where the action of $g - 1$ on M is given by $x' + y'z' = x$. □

Using Lemma 3.1 we prove the following generalization of Lemma 6.4 in [**SFB**].

Theorem 3.2. *Let $x, x_1, \ldots, x_s$ be pairwise commuting nilpotent operators on $M = k^d$ such that the nilpotencies of $x, x_1, \ldots, x_s$ are $p^n, p^{n_1}, \ldots, p^{n_s}$ respectively, where $n \geq n_1 \geq n_2 \geq \cdots n_s$, and k is a field of characteristic p. Let $f \in k[t_1, \ldots, t_s]$ be a polynomial having no constant or linear terms and $\widetilde{x} = x + f(x_1, \ldots, x_s)$. Then M is free as a $k[C_{p^{m_0}}]$-module where the action of $g - 1$ on M is given by x if and only if M is free as a $k[C_{p^{m_0}}]$-module where the action of $g - 1$ on M is given by $\widetilde{x}$, for a generator g of $C_{p^{m_0}}$.*

Proof. Note that $f(x_1, \ldots, x_s) = \sum_{j=1}^{s} x_j y_j$ where $y_j = f_j(x_1, \ldots, x_s)$ for some $f_j \in k[t_1, \ldots, t_s]$ having no constant term, for $j = 1, \ldots, s$, and also y_j is nilpotent. First apply Lemma 3.1 to the triple x, x_1, y_1, then to the triple $x + x_1y_1, x_2, y_2$, and finally to the triple $x + x_1y_1 + \cdots + x_{s-1}y_{s-1}, x_s, y_s$. Then the statement follows. □

As noted in the introduction Main Theorem is a restatement of Theorem 3.2.

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