Arch. Math. 76 (2001) 25–29
0003-889X/01/010025-05 $ 2.50/0

Archiv der Mathematik

A note on a theorem of Dwyer and Wilkerson

By

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Abstract. We prove a version of Theorem 2.3 in [1] for the non-elementary abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$, $n \geqq 2$. Roughly, we describe the equivariant cohomology of (union of) fixed point sets as the unstable part of the equivariant cohomology of the space localized with respect to suitable elements of the cohomology ring of $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$.

Introduction. There are many results in the literature for structures related to elementary abelian p-groups that fail for abelian p-groups. This article arose in finding out the use of being elementary in Theorem 2.3 in [1]. Mainly we use their terminology and notation in this article; some of the exceptions are that G denotes an abelian p-group, E denotes an elementary abelian p-group, and H^*_G denotes $H^*(BG)$. The coefficients are in $\mathbb{F}_p$ for any cohomology. Dwyer and Wilkerson used Hsiang's generalization of what they call "localization theorem of Borel-Atiyah-Segal-Quillen" to prove that the unstable part $\mathcal{U}n(S_K^{-1}H^*_E(X))$ of the $\mathcal{A}_p$—$S_K^{-1}H^*_E$-module $S_K^{-1}H^*_E(X)$ is isomorphic to $H^*_E(X^K)$, where $K \leqq E$ and $\mathcal{A}_p$ is the $\mathbb{F}_p$-algebra generated by mod p Steenrod operations. We observe that some of their lemmas can be modified for the non-elementary abelian group $G_n = \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$, $n \geqq 2$, and obtain an analogue of their theorem. Recall that an action on a G-space X is called *semi-free* if the isotropy subgroup G_x is G or 1 for all $x \in X$.

Theorem. *For $n \geqq 2$, let $G_n = \langle e, g \mid e^2 = g^{2^n} = 1, eg = ge\rangle > E = \langle e, g^{2^{n-1}}\rangle$ with the cohomology groups $H^*_{G_n} = \mathbb{F}_2[t_1, \theta_n] \otimes \Lambda(v_n)$ and $H^*_E = \mathbb{F}_2[t_1, t_n]$, and let X be a finite G_n-CW-complex. In parts* (iii) *and* (iv) *assume also at least one $\mathbb{Z}_{2^n}$-subgroup of G_n acts semi-freely on X. Then*

(i) $H^*_{G_n}(X^{eg^{2^{n-1}}}, X^E) \xrightarrow{\cong} \mathcal{U}n\left(H^*_{G_n}(X, X^E)\left[\dfrac{1}{t_1^2\theta_n}\right]\right)$,

(ii) $H^*_{G_n}(X^e \cup X^{eg^{2^{n-1}}}, X^E) \xrightarrow{\cong} \mathcal{U}n\left(H^*_{G_n}(X, X^E)\left[\dfrac{1}{t_1}\right]\right)$,

(iii) $H^*_{G_n}(X^{g^{2^{n-1}}} \cup X^{eg^{2^{n-1}}}, X^E) \xrightarrow{\cong} \mathcal{U}n\left(H^*_{G_n}(X, X^E)\left[\dfrac{1}{\theta_n}\right]\right)$,

(iv) $H^*_{G_n}(X^{g^{2^{n-1}}} \cup X^e, X^E) \xrightarrow{\cong} \mathcal{U}n\left(H^*_{G_n}(X, X^E)\left[\dfrac{1}{t_1^2 + \theta_n}\right]\right)$.

Mathematics Subject Classification (1991): Primary 55M35; Secondary 54E30.

Preliminaries. As in [2] for a multiplicative set $S \supset \{1\}$ contained in the center of H_G^* and a G-CW-complex X, define $X^S = \{\, x \in X : \ker(\mathrm{res}_{G,G_x} H_G^* \longrightarrow H_{G_x}^*)\} \cap S = \emptyset \,\}$, where G_x is the isotropy subgroup of x. For $K \leqq E$, let S_K be the multiplicative subset of $H_E^* \backslash \sqrt{(0)}$ generated by $\beta(u)$ where $u \in H_E^1$ and restricts nontrivially to H_K^*. Note that X^a is determined by where the superscript a lies in, either a is in H_G^* or in G. For $g \in G$, $X^{\langle g \rangle} = X^g$ so we write X^g, for $r \in H_G^*$, $S_r = \{r^i : i \geqq 0\}$ and we write X^r for X^{S_r}.

Proposition 1. *Let G be a compact Lie group, X a compact G-space, and $Y \subseteqq X$ be G-invariant subspace. If $S \subset H_G^*$ is a multiplicative system and $i : (X^S, Y^S) \hookrightarrow (X, Y)$ is the inclusion, then the localized restriction homomorphism $\rho^{-1} = S^{-1} i^* : S^{-1} H_G^*(X, Y) \longrightarrow S^{-1} H_G^*(X^S, Y^S)$ is an isomorphism.*

Proof. Since localization is an exact functor, the long exact sequence of a pair in cohomology remains exact after localizing with respect to S. Then use Theorem III.1 in [2] and the 5-Lemma to obtain the required isomorphism. $\square$

Let $M = \oplus_{i \geqq 0} M^i$ be an $\mathscr{A}_2$-module where M^i denotes elements of degree i in M. M is called *unstable* if the dimension axiom is satisfied, *i.e.*, if $\mathrm{Sq}^i \cdot m = \mathrm{Sq}^i(m) = 0$ for $i > \dim(m)$ and for all homogeneous $m \in M$. The largest unstable submodule of M is denoted by $\mathscr{U}n(M)$.

Let R be the unstable $\mathscr{A}_2$-module $H_{\mathbb{Z}_4}^* = \mathbb{F}_2[\tau] \otimes \Lambda(u)$, where the polynomial generator τ is in R^2 and the exterior algebra generator u is in R^1. The $\mathscr{A}_2$-module structure of R induces an action on the localization $S_\tau^{-1} R = R\left[\dfrac{1}{\tau}\right]$ of R so that $R\left[\dfrac{1}{\tau}\right]$ is an $\mathscr{A}_2$—$R\left[\dfrac{1}{\tau}\right]$-module which is no longer unstable as an $\mathscr{A}_2$-module. Note that for $i \geqq 0$, $\mathrm{Sq}^i\left(\dfrac{1}{\tau}\right) = 0$ if i is odd and $\mathrm{Sq}^i\left(\dfrac{1}{\tau}\right) = \tau^{(i/2)-1}$ if i is even.

Let $R_n = H_{\mathbb{Z}_{2^n}}^* = \mathbb{F}_2[\theta_n] \otimes \Lambda(u_n)$ for $n > 2$ and $R_2 = H_{\mathbb{Z}_4}^* = \mathbb{F}_2[\tau] \otimes \Lambda(u)$ to simplify notation. First we recall a well known result which makes it possible to reduce the problem to a simpler case. It can be verified using the Bockstein operator β induced by the short exact coefficient sequence $0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.

Lemma 2. *$R_n \cong R_2$ as $\mathscr{A}_2$-algebras for all $n \geqq 2$.*

Lemma 3. *Let N be an unstable $\mathscr{A}_2$-module. Then $\eta : R_2 \otimes N \longrightarrow \mathscr{U}n\left((R_2 \otimes N)\left[\dfrac{1}{\tau}\right]\right)$ defined by $\eta(x) = \dfrac{x}{1}$ is an isomorphism of algebras and $\mathscr{A}_2$-modules.*

Proof. We will use R for R_2.

Assertion. If $\frac{z}{\tau} \in \mathscr{U}n\left((R \otimes N)\left[\dfrac{1}{\tau}\right]\right)$ for $z \in R \otimes N$, then there exists $y \in R \otimes N$ such that $\dfrac{z}{\tau} = \dfrac{y}{1}$. Suppose the assertion is true. If $\dfrac{x}{\tau^l} \in \mathscr{U}n\left((R \otimes N)\left[\dfrac{1}{\tau}\right]\right)$, then $\dfrac{x}{\tau^{l-1}}, \dfrac{x}{\tau^{l-2}}, \ldots, \dfrac{x}{\tau} \in \mathscr{U}n\left((R \otimes N)\left[\dfrac{1}{\tau}\right]\right)$. By applying the assertion we first obtain y_1, then y_2, and $y_3, \ldots, y_l$ in $R \otimes N$ such that $\dfrac{x}{\tau} = \dfrac{y_1}{1}$, $\dfrac{x}{\tau^2} = \dfrac{y_1}{\tau} = \dfrac{y_2}{1}, \ldots, \dfrac{x}{\tau^l} = \dfrac{y_{l-1}}{1} = \eta(y_l)$. Hence η is surjective. Injectivity follows from the fact that τ is not a zero divisor.

Proof of the assertion. Let $\frac{z}{\tau} \in \mathcal{U}n\left((R\otimes N)\left[\frac{1}{\tau}\right]\right)$, where $z \in R\otimes N$. There are $d, q \in \mathbb{N}$, $n_i, m_i \in N$ such that $z = \tau^d \otimes n_d + \cdots + \tau \otimes n_1 + 1 \otimes n_0 + u\tau^q \otimes m_q + u\tau^{q-1} \otimes m_{q-1} + \cdots + u\tau \otimes m_1 + u \otimes m_0$ where nonzero n_i's are $\mathbb{F}_2$-linearly independent, so are the nonzero m_i's. Then

$$\frac{z}{\tau} - \left(\frac{\tau^{d-1}}{1} \otimes n_d + \cdots + \frac{\tau}{1} \otimes n_2 + \frac{1}{1} \otimes n_1 + \frac{u\tau^{q-1}}{1} \otimes m_q + \cdots + \frac{u}{1} \otimes m_1\right)$$
$$= \frac{1}{\tau} \otimes n_0 + \frac{u}{\tau} \otimes m_0$$

is unstable as a sum of finitely many unstable elements. Therefore $\mathrm{Sq}^k\left(\frac{1}{\tau} \otimes n_0 + \frac{u}{\tau} \otimes m_0\right) = \frac{0}{1}$ for $k > \dim\left(\frac{z}{\tau}\right)$, hence $\mathrm{Sq}^k\left(\frac{1}{\tau} \otimes n_0\right) = \mathrm{Sq}^k\left(\frac{u}{\tau} \otimes m_0\right) = u\,\mathrm{Sq}^k\left(\frac{1}{\tau} \otimes m_0\right)$ in $(R\otimes N)\left[\frac{1}{\tau}\right]$ and $\mathrm{Sq}^k\left(\frac{u}{\tau} \otimes n_0\right) = u\,\mathrm{Sq}^k\left(\frac{1}{\tau} \otimes n_0\right) = u\left(u\,\mathrm{Sq}^k\left(\frac{1}{\tau} \otimes m_0\right)\right) = \frac{0}{1}$. In particular for an even $k > \dim\left(\frac{z}{\tau}\right)$, we have

$$\mathrm{Sq}^k\left(\frac{u}{\tau} \otimes n_0\right) = \frac{u}{\tau} \otimes \mathrm{Sq}^k(n_0) + u \otimes \mathrm{Sq}^{k-2}(n_0)$$
$$+ u\tau \otimes \mathrm{Sq}^{k-4}(n_0) + \cdots + u\tau^{k/2-2} \otimes \mathrm{Sq}^2(n_0) + u\tau^{k/2-1} \otimes n_0 = \frac{0}{1}.$$

Then there is an $r \geqq 0$ such that

$$\tau^{r-1} \otimes \mathrm{Sq}^k(n_0) + \tau^r u \otimes \mathrm{Sq}^{k-2}(n_0) + u\tau^{r+1} \otimes \mathrm{Sq}^{k-4}(n_0) + \cdots$$
$$+ u\tau^{k/2-2+r} \otimes \mathrm{Sq}^2(n_0) + u\tau^{k/2-1+r} \otimes n_0 = 0\,.$$

Since the sets $\{\mathrm{Sq}^i(n_0) \mid i \in \mathbb{N}, \mathrm{Sq}^i(n_0) \neq 0\}$ and $\{u\tau^j \mid j \in \mathbb{N}\}$ consist of $\mathbb{F}_2$-linearly independent elements, the previous equation implies $n_0 = 0$ which in turn implies $\mathrm{Sq}^k\left(\frac{u}{\tau} \otimes m_0\right) = \frac{0}{1}$. Using this equation in similar arguments as for n_0 we obtain $m_0 = 0$. Therefore $z = \tau y$ and $\eta(y) = \frac{z}{\tau}$, where

$$y = \tau^{d-1} \otimes n_d + \cdots + \tau \otimes n_2 + 1 \otimes n_1 + u\tau^{q-1} \otimes m_q + \cdots + u\tau \otimes m_2$$
$$+ u \otimes m_1 \in R \otimes N. \quad \square$$

When $G = \mathbb{Z}_{2^n}$, $n \geqq 2$, one cannot capture the fixed point sets of various subgroups of G in a G-space X by localizations. For there is only one polynomial generator θ_n in $H_G^* = k[\theta_n] \otimes \Lambda(v_n)$ which determines only $X^{\mathbb{Z}_2} = X^{\theta_n}$. So the assumption that G acts semi-freely on X is natural in the following.

Proposition 4. *Suppose that $\mathbb{Z}_{2^n}$ acts semi-freely on a $\mathbb{Z}_{2^n}$-CW-complex X. Then we have*

$$\mathcal{U}n\left(H^*_{\mathbb{Z}_{2^n}}(X)\left[\frac{1}{\theta_n}\right]\right) \cong H^*_{\mathbb{Z}_{2^n}}(X^{\mathbb{Z}_{2^n}}) \text{ and } H^*(X^{\mathbb{Z}_{2^n}}) \cong \mathbb{F}_2 \otimes_{H^*_{\mathbb{Z}_{2^n}}} \mathcal{U}n\left(H^*_{\mathbb{Z}_{2^n}}(X)\left[\frac{1}{\theta_n}\right]\right).$$

Proof. The hypothesis implies that $X^{\mathbb{Z}_2} = X^{\mathbb{Z}_{2^n}}$. Then the first isomorphism follows from Lemma 3, because $H^*_{\mathbb{Z}_{2^n}}(X)\left[\frac{1}{\theta_n}\right] \cong H^*_{\mathbb{Z}_{2^n}}(X^{\theta_n})\left[\frac{1}{\theta_n}\right]$ by the localization theorem and

$X^{\theta_n} = X^{\mathbb{Z}_2}$, where $\mathbb{Z}_2 \leqq \mathbb{Z}_{2^n}$. The second isomorphism is true because $H^*_{\mathbb{Z}_{2^n}}(X^{\mathbb{Z}_{2^n}}) \cong H^*(X^{\mathbb{Z}_{2^n}}) \otimes H^*_{\mathbb{Z}_{2^n}}$. $\square$

Proof of the Theorem. For $G_n = \langle e, g \mid e^2 = g^{2^n} = 1, eg = ge\rangle$ with $n \geqq 2$, there is a unique maximal elementary abelian subgroup $E = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e, g^{2^{n-1}}\rangle$ which contains all $\mathbb{Z}_2$-subgroups of G_n, namely, $\langle e\rangle$, $\langle eg^{2^{n-1}}\rangle$, and $\langle g^{2^{n-1}}\rangle$. Thus the singular set of a G_n-space X is $\mathscr{S}_{G_n}(X) = X^e \cup X^{eg^{2^{n-1}}} \cup X^{g^{2^{n-1}}}$, where any two of the sets on the right intersect at X^E. If $\langle g\rangle$ (similarly $\langle eg\rangle$) acts semi-freely on X, then $X^{g^{2^{n-1}}} = X^g$ and $X^{G_n} = X^e \cap X^g = X^e \cap X^{g^{2^{n-1}}} = X^E$; in addition since $X^{g^{2^{n-1}}} \supset X^{eg}$ we obtain $X^g \supset X^{eg}$; hence $X^{eg} = X^g \cap X^{eg} = X^{G_n}$. By Lemma 2, $H^*_{G_n}$ is isomorphic to $H^*_{\mathbb{Z}_2 \times \mathbb{Z}_4} \cong H^*_{\mathbb{Z}_2} \otimes H^*_{\mathbb{Z}_4} := \mathbb{F}_2[t_1] \otimes R_2$ as $\mathscr{A}_2$-algebras. Set $H^*_E \cong \mathbb{F}_2[t_1, t_n]$. Then $\mathrm{res}_{G_n,E}(t_1) = t_1$, $\mathrm{res}_{G,E}(\theta_n) = t_n^2$, $\mathrm{res}_{G,E}(v_n) = 0$; and $X^{t_1} = X^e \cup X^{eg^{2^{n-1}}}$, $X^{\theta_n} = X^{eg^{2^{n-1}}} \cup X^{g^{2^{n-1}}}$, $X^{t_1^2\theta_n} = X^{eg^{2^{n-1}}}$, moreover $X^{T_2} = X^E$, where $T_2 = t_1^2\theta_n(t_1^2 + \theta_n)$. Therefore $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ behaves exactly like $\mathbb{Z}_2 \times \mathbb{Z}_4$. So it is sufficient to prove the Theorem for $n = 2$.

Theorem′. *Let $G = \langle e, f \mid e^2 = f^4 = 1, ef = fe\rangle > E = \langle e, f^2\rangle$ and X be a finite G-CW-complex. In parts* (iii) *and* (iv) *assume also at least one $\mathbb{Z}_4$-subgroup of G acts semi-freely on X. Then*

(i) $H^*_G(X^{ef^2}, X^E) \xrightarrow{\cong} \mathscr{U}n\left(H^*_G(X, X^E)\left[\dfrac{1}{t_1^2\tau}\right]\right)$,

(ii) $H^*_G(X^e \cup X^{ef^2}, X^E) \xrightarrow{\cong} \mathscr{U}n\left(H^*_G(X, X^E)\left[\dfrac{1}{t_1}\right]\right)$,

(iii) $H^*_G(X^{f^2} \cup X^{ef^2}, X^E) \xrightarrow{\cong} \mathscr{U}n\left(H^*_G(X, X^E)\left[\dfrac{1}{\tau}\right]\right)$,

(iv) $H^*_G(X^{f^2} \cup X^e, X^E) \xrightarrow{\cong} \mathscr{U}n\left(H^*_G(X, X^E)\left[\dfrac{1}{t_1^2 + \tau}\right]\right)$.

Proof. For r, s in a multiplicative system $S \subset H^*_G$ we have $X^{rs} = X^r \cap X^s$. In particular, $X^{t_1^2\tau} = X^{ef^2}$, $X^{t_1} = X^e \cup X^{ef^2}$, $X^\tau = X^{f^2} \cup X^{ef^2}$ and $X^{t_1^2+\tau} = X^{f^2} \cup X^e$. Moreover $X^e \cap X^{ef^2} = X^e \cap X^{f^2} = X^E$ and $(X^E)^r = X^E$ for $r \in S$. If $X^r = X^A \cup X^B$ for some subgroups A, B of G with $X^A \cap X^B = X^E$, then $H^*_G(X^r, X^E) = H^*_G(X^A, X^E) \oplus H^*_G(X^B, X^E)$. Also when $G \cong A \times G/A$ for some $A \leqq G$ with $X^A \supseteqq X^E$, we have $H^*_G(X^A, X^E) \cong H^*_{G/A}(X^A, X^E) \otimes H^*_A$. Note that $H^*_{G/A}(X^A, X^E)$ is a trivial H^*_A-module. We can write $G \cong \langle e\rangle \times \langle f\rangle \cong \langle ef^2\rangle \times \langle f\rangle \cong \langle ef^2\rangle \times \langle ef\rangle$. However $G \ncong \langle f^2\rangle \times K$ for any $K < G$ because f^2 is a non-generator of G as $\langle f^2\rangle$ is the Frattini subgroup of G. Therefore for parts (iii) and (iv) we will make use of the hypothesis $X^{f^2} = X^{\mathbb{Z}_4}$ in order to write $H^*_G(X^{f^2}, X^E) = H^*_G(X^A, X^E) \cong H^*_{G/A}(X^A, X^E) \otimes H^*_A$, where $A = \mathbb{Z}_4$ is the $\mathbb{Z}_4$-subgroup of G acting semi-freely on X.

By Proposition 1, for $r \in S$, the inclusion $(X^r, X^E) \hookrightarrow (X, X^E)$ induces an isomorphism in the localized equivariant cohomology rings $H^*_G(X, X^E)\left[\dfrac{1}{r}\right] \cong H^*_G(X^r, X^E)\left[\dfrac{1}{r}\right]$. In particular,

(i) $H^*_G(X, X^E)\left[\dfrac{1}{t_1^2\tau}\right] \cong H^*_{G/\langle ef^2\rangle}(X^{ef^2}, X^E) \otimes \left(H^*_{\langle ef^2\rangle}\left[\dfrac{1}{t_3^4}\right]\right)$,

(ii) $H^*_G(X, X^E)\left[\dfrac{1}{t_1}\right] \cong H^*_{G/\langle e\rangle}(X^e, X^E) \otimes \left(H^*_{\langle e\rangle}\left[\dfrac{1}{t_1}\right]\right) \oplus H^*_{G/\langle ef^2\rangle}(X^{ef^2}, X^E) \otimes \left(H^*_{\langle ef^2\rangle}\left[\dfrac{1}{t_3}\right]\right)$,

(iii) $H^*_G(X, X^E)\left[\dfrac{1}{\tau}\right] \cong H^*_{G/\mathbb{Z}_4}(X^{\mathbb{Z}_4}, X^E) \otimes \left(H^*_{\mathbb{Z}_4}\left[\dfrac{1}{\tau}\right]\right) \oplus H^*_{G/\langle ef^2\rangle}(X^{ef^2}, X^E) \otimes \left(H^*_{\langle ef^2\rangle}\left[\dfrac{1}{t_3^2}\right]\right),$

(iv) $H^*_G(X, X^E)\left[\dfrac{1}{t_1^2+\tau}\right] \cong H^*_{G/\mathbb{Z}_4}(X^{\mathbb{Z}_4}, X^E) \otimes \left(H^*_{\mathbb{Z}_4}\left[\dfrac{1}{\tau}\right]\right) \oplus H^*_{G/\langle e\rangle}(X^{e}, X^E) \otimes \left(H^*_{\langle e\rangle}\left[\dfrac{1}{t_1^2}\right]\right).$

Also, note that $\mathcal{U}n(N \otimes S^{-1}R) \cong \mathcal{U}n(N \otimes T^{-1}R)$ for any non-trivial multiplicative subsets S and T in R with $S \supseteqq T$. Therefore it remains to show that $\mathcal{U}n\left((N \otimes H^*_A)\left[\dfrac{1}{r}\right]\right) \cong N \otimes H^*_A$, where N is an unstable $\mathcal{A}_2 - H^*_A$-module which is trivial as an H^*_A-module and r is the polynomial generator in the cohomology ring H^*_A. We have $r \in H^1_A$ if $A \cong \mathbb{Z}_2$, $r \in H^2_A$ if $A \cong \mathbb{Z}_4$. Hence the desired isomorphisms are true by Proposition 3.6 in [1] when $A \cong \mathbb{Z}_2$, and by Lemma 3 when $A \cong \mathbb{Z}_4$. $\square$

This work represents a part of the author's Ph.D. thesis completed under the supervision of Professor Amir H. Assadi. The author wishes to thank Professor A. H. Assadi for useful suggestions and comments.

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Eingegangen am 7. 5. 1999 *)

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*) Eine überarbeitete Fassung ging am 13. 4. 2000 ein.