Last revised May 9, 2011

CHAPTER 9

Statistical Modelling

The previous chapters were concerned with basic mechanisms and theory that govern the propagation of radio waves in space. In those discussions, the propagation medium is considered to be a time invariant channel. This is not true in real life and there are many physical phenomena that cause variations in the received signal. The variations of meteorological conditions, motion of the scatterers, solar activities are among the main reasons of such variations.

In many cases, even though the medium is time invariant, the radio waves are also subject to distortion. A rectangular transmitted pulse will have a distorted shape at receiver due to the change in the velocity of propagation with frequency.

Such effects depend on too many parameters and a simple theoretical model is not available to predict the results. Instead, some empirical models are developed either by fitting curves to measured data or by statistical and analytical studies on model problems. Our purpose in this chapter is to understand such phenomena and methods used to analyze and alleviate such problems.

9.1. Fading

In many cases the total signal at the receiver is a sum of several rays. As the medium changes with time, the distance covered by various rays changes continuously which results in a change in the phase of each ray component. When the waves are in phase they will add and otherwise they will tend to cancel each other. As a result the signal amplitude and phase will change in a random manner. This is called *fading*. The fading may vary with time, geographical position and/or radio frequency, and the attenuation factor is modelled as a random process.

9.1.1. Fading in Time. Fading may either be *fast* or *slow* depending on the rate at which the amplitude and phase of the signal changes at a single location in space. The *coherence time* of a signal is defined as the minimum time required for the signal to become uncorrelated from its previous value. Consider a random signal $s(t)$ with mean $\mu(t)$ and variance $\sigma^2(t)$ which may be functions of time. The auto-correlation function of s is defined as

$$R_s(t_1, t_2) = E\{s(t_1)\, s^*(t_2)\} \tag{9.1.1}$$

where $E\{\cdot\}$ denotes the ensemble average operator. If the process $s(t)$ is wide sense stationary, its mean and variance are independent of time, and the auto-correlation depends only on the difference $\tau = t_1 - t_2$. The auto-correlation function is Hermitian symmetric, i.e., $R_s(\tau) = R_s^*(-\tau)$ and therefore $R_s(0)$ is real. Furthermore, it satisfies the property

$$|R_s(\tau)| \leq R_s(0). \tag{9.1.2}$$

For a practical signal, the value of τ for which $|R_s(\tau)|$ decays to a small fraction of $R_s(0)$ is defined as the coherence time of the signal. The rate of fading is determined by comparing the coherence time of the signal to the rate of change of the channel.

Slow fading: occurs when the change of rate of the channel characteristics are slow as compared to the coherence time of the signal. Slow fading is typically caused by shadowing, where a large obstruction obscures the main signal path. The amplitude change caused by shadowing is often modeled using a log-normal distribution with a standard deviation according to the log-distance path loss model.

Fast fading: occurs if the channel changes much faster than the coherence time of the signal.

9.1.2. Fading in Space. With a fixed transmitter, if we measure the signal strength at different locations, we will observe that the signal strength changes as a function of location. We define the cross-correlation between the signals at two different locations as

$$\rho(\mathbf{r}_1, \mathbf{r}_2) = E\{s_1(t)\, s_2^*(t)\} \tag{9.1.3}$$

where $E\{\cdot\}$ denotes the time average operator, and $s_1(t)$ and $s_2(t)$ denote the signal at positions $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. If the process is wide sense stationary in space, the spatial cross-correlation is a function of distance $l = |\mathbf{r}_1 - \mathbf{r}_2|$ between the two locations. As in the case of auto-correlation function, $\rho(0)$ is real, and $|\rho(l)|$ has its maximum at $l = 0$. If we use two antennas located at different positions, the signals received by the antennas will become *practically* uncorrelated as the distance between them is increased. We may arbitrarily set a level for $|\rho(l)|$ below which we say that the signals are uncorrelated. The corresponding length l can be defined as correlation length.

The scale of fading for is determined by comparing the variation rate in space (correlation length) to the wavelength. It is possible to identify three fading scales.

Small scale fading: is the variations of signal in a distance about half the wavelength and is due to multipath propagation in the vicinity of the mobile or the base-station. The statistical distribution of the signal measured on this scale is typically found to be Ricean, if a line-of-sight path exists, or Rayleigh otherwise.

Middle scale fading: is observed when the received signal is averaged over a sector whose dimensions are of the order of $10\lambda_0$ or so. The signal so obtained is known as the local mean signal. The local mean can also be considered as a random variable which is a function of position. The variations in the local mean signal is the middle scale fading. These variations are caused by the random variations in the environment such as building heights, gaps between buildings, differences in building design and construction materials, and the presence of trees and other moving objects. The signal undergoes many reflections and/or refractions each of which appear as a multiplicative term in the attenuation factor. Since these effects are independent, the resulting attenuation factor is the product of many independent random terms. This gives rise to log-normal statistics from the central limit theorem.

Large scale fading: is due to path dependence of the received signal and describes the gross variations of the area mean with distance from the transmitter. This scale of fading is due to the spreading of the wave front in space over the given environment. The fading in this scale is commonly described by the general properties of the propagation path. Obviously, the long term average of the signal at a constant distance from the transmitter will be different in rural, urban, mountainous, sea, etc. environments.

9.1.3. Fading in Frequency. As the carrier frequency of a signal is varied, the magnitude of the change in amplitude will vary. This is basically due to the dependence of phase on frequency. The *coherence bandwidth* measures the separation in frequency after which two signals will experience uncorrelated fading.

Flat fading: occurs if the coherence bandwidth of the channel is larger than the bandwidth of the signal. In this case, all frequency components of the signal will experience the same magnitude of fading. In this case the signal amplitude is decreased but the signal will not be distorted.

Frequency selective fading: occurs if the coherence bandwidth of the channel is smaller than the bandwidth of the signal. Different frequency components of the signal therefore experience de-correlated fading. This will cause distortion in the signal.

One may think that a frequency selective fading channel is more problematic than the flat fading channel. However, in a flat fading channel, when a deep fade occurs the signal is completely lost. On the other hand, it is highly unlikely that all parts of the signal will be simultaneously affected by a deep fade in a frequency selective fading channel, since different frequency components of the signal are affected independently. This may be exploited in certain modulation schemes such as OFDM and CDMA.

9.2. Statistical Description

When the signal strength changes with time (or in space), we need to introduce suitable definitions for the average signal level and the variations of the signal strength. Most commonly, the average signal level is described by the *median* signal strength, which is the level that is exceeded half of the time. For fading in space, this level becomes the level that is exceeded at half of the locations of interest. Although it gives an idea about the average level, the median signal does not show how deep the fading is. Two signals having the same median signal strength may be subject to fading with different depths. One way to describe the fading depth is to specify one or more levels which are exceeded during a given percentage of time. We will us the notation $E(q)$ to denote the level that is exceeded during $q\%$ of the time (or at $q\%$ of the locations). A complete description can only be done by specifying $E(q)$ for all q values between 0 and 100. This is equivalent to specifying the cumulative distribution function (cdf). An equivalent description of the field would be to give the probability density function (pdf) of the random value of the signal strength. However, this is not always possible in practice. The fading depth is generally measured as the difference of $E(10)$ and $E(90)$, i.e.,

$$\text{Fading depth} = E(10) - E(90). \tag{9.2.1}$$

Since the signal strength is a random process, specifying its probability distribution is not enough. One must also specify the rate of change of signals in time. Typically this is done by giving the number of crossings per unit time. Again, this single parameter does not completely describe the process. Generally, instead of giving a complete description, only the second order statistics of the process, i.e., its auto-correlation function is specified.

There are several probability distributions that are commonly used to describe the fading. The most commonly encountered probability distribution is the normal distribution. The reason is that, most of the physical phenomena is a result of several independent random effects. The central limit theorem then tells us that the resulting process will have a normal distribution. As mentioned previously, in many cases the field at a receiver location is the sum of different waves arriving at the receiver through different paths. The total field can be written in time domain as

$$\mathcal{E}(t) = \sum_{n=1}^{N} \mathcal{E}_n(t) \tag{9.2.2}$$

where

$$\mathcal{E}_n(t) = A_n \cos[\omega_c t + \mathbf{k}_n \cdot \mathbf{r} + \psi_n] \tag{9.2.3}$$

where ω_c is the carrier frequency, $\mathbf{k}_n$ is the wave vector, A_n is the amplitude, and ψ_n is the phase of the nth wave. After some trigonometric manipulations, the total signal can be written as

$$\mathcal{E}(t) = \mathrm{Re}\left\{[I(t) + jQ(t)]\, e^{j\omega_c t}\right\} = I(t)\cos\omega_c t - Q(t)\sin\omega t \tag{9.2.4}$$

where $I(t)$ and $Q(t)$ are the in-phase and quadrature components, respectively, and can be written as

$$I(t) = \sum_{n=1}^{N} A_n \cos(\omega_c t + \alpha_n), \tag{9.2.5}$$

$$Q(t) = \sum_{n=1}^{N} A_n \sin(\omega_c t + \alpha_n) \tag{9.2.6}$$

where $\alpha_n = \mathbf{k}_n \cdot \mathbf{r} + \psi_n$. In this derivation, we have ignored any Doppler shift that may be caused due to moving scatterers. The phase angle ψ_n may have any value between zero and 2π and is generally assumed to be uniformly distributed. If N is sufficiently large, then by central limit theorem $I(t)$ and $Q(t)$ are independent Gaussian random processes which are completely characterized by their mean value and auto-correlation function. When ψ_n is uniformly distributed, the mean values of the in-phase and quadrature components are both zero which, in turn, implies that $\mathcal{E}(t)$ is also zero mean:

$$\langle I(t)\rangle = \langle Q(t)\rangle = \langle \mathcal{E}(t)\rangle = 0 \tag{9.2.7}$$

where $\langle\cdot\rangle$ denotes ensemble average. Furthermore, it can be shown that $\langle I^2(t)\rangle = \langle Q^2(t)\rangle$ and $\langle I(t)\, Q(t)\rangle = 0$. The envelope of the received total electric field is $|\mathcal{E}(t)| = \sqrt{I^2(t) + Q^2(t)}$ and at a fixed time instant, it will have a Rayleigh distribution.

The pdf of a real random variable Z with normal distribution is given by

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right) \tag{9.2.8}$$

where μ, and σ^2 are the mean and variance of Z. The pdf of a Rayleigh distributed random variable R is

$$p(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) u(r) \tag{9.2.9}$$

where $u(r)$ is the unit step function. The mean and variance of R are given by

$$\langle R \rangle = \int_0^\infty \frac{r^2}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr = \sqrt{\frac{\pi}{2}}\sigma, \tag{9.2.10}$$

$$\langle R^2 \rangle = \int_0^\infty \frac{r^3}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr = 2\sigma^2. \tag{9.2.11}$$

It is customary to choose the variances of $I(t)$ and $Q(t)$ as $\sigma^2/2$. Then the variance of the envelope is σ^2 which is the mean power of the received field.

The above derivation corresponds to a propagation path which does not have a direct path. When a direct path exists, the phase of the direct path will not be a random variable, instead it will be a constant. The mean value of the direct path signal will no more be zero and the total field will have a Ricean distribution (sometimes called a non-central Rayleigh distribution). The pdf of a Ricean distributed random variable R with $\langle R^2 \rangle = \sigma^2 + \nu^2$ is

$$p(r) = \frac{2r}{\sigma^2} \exp\left(-\frac{r^2 + \nu^2}{\sigma^2}\right) I_0\left(\frac{2r\nu}{\sigma^2}\right). \tag{9.2.12}$$

Another commonly encountered distribution in propagation is the log-normal distribution. A random variable whose logarithm is normally distributed is said to have a log-normal distribution. If X is a random variable with a normal distribution, then $Y = \exp(X)$ has a log-normal distribution; likewise, if Y is log-normally distributed, then $X = \log(Y)$ is normally distributed. This occurs when a single ray is affected by many independent multiplicative factors. For example, if a ray undergoes several reflections before arriving the receiver, the received signal will have a log-normal distribution, since the reflection coefficient at each reflection point can be considered independent random variables. The pdf of a random variable R with log-normal distribution is

$$p(r) = \frac{1}{r\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln r - \mu)^2}{2\sigma^2}\right) u(r). \tag{9.2.13}$$

For log-normal distribution the raw moments are

$$\langle R \rangle = e^{\frac{1}{2}\sigma^2 + \mu}; \quad \langle R^2 \rangle = e^{2\mu + 2\sigma^2}. \tag{9.2.14}$$

All the above distributions have at most two parameters. Therefore, specifying the median signal level and fading depth will completely characterize the distribution of the signal at a given time instant. For example, for Rayleigh fading which has a single parameter, if the median level is $E(50) = E_m$ we can write

$$\int_0^{E_m} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr = 1 - \exp\left(-\frac{E_m^2}{2\sigma^2}\right) = 0.5 \tag{9.2.15}$$

which can be solved for σ^2 to give

$$\sigma^2 = \frac{E_m^2}{2\ln 2}. \tag{9.2.16}$$

The $E(q)$ values can be written similarly as

$$\int_0^{E(q)} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr = 1 - \exp\left(-\frac{E^2(q)}{2\sigma^2}\right) = 1 - q \tag{9.2.17}$$

and we find

$$E(q) = E_m \sqrt{-\frac{\ln q}{\ln 2}} = E_m \sqrt{-\frac{\log_{10} q}{\log_{10} 2}}. \tag{9.2.18}$$

Then, the fading depth can be calculated as

$$F_d = E(10) - E(90) = E_m \left(\sqrt{\frac{1}{\log_{10} 2}} - \sqrt{\frac{\log_{10}(10/9)}{\log_{10} 2}}\right) = 1.433 E_m. \tag{9.2.19}$$

For a Rayleigh fading channel, (9.2.18) shows that, if we want to secure communication for $q\%$ of the time, we should increase the field strength by a factor of $\sqrt{-\log_{10} q/\log_{10} 2}$ above the median level.

9.3. Empirical Path Loss Models

In many cases, it is very difficult to obtain a simple expression for path loss by using theoretical and analytical derivations. For example, in urban areas, there are too many scatterers that affect the propagation. Furthermore, many of these scatterers are in constant motion. In such cases empirical models, which are themselves based on measurements performed in similar environments and/or on theoretical models are used. In the next few sections we will present some empirical models that are useful in predicting the median signal strength for various environments.

9.3.1. Okumura-Hata Model. The Okumura–Hata model is a fully empirical prediction method, [**62**] that is commonly used in mobile systems, i.e., a propagation path consisting of a fixed base station and a mobile receiver. This model is based entirely on an extensive series of measurements made in and around Tokyo city over the frequency interval 200 MHz to 2 GHz. There is no attempt to base the predictions on a physical model. Originally, the prediction was done by using a set of graphs. Later most important graphs were expressed by approximate formulas, [**63**].

The method divides the prediction area into three different clutter and terrain categories. These are:

Open areas: that contain no tall trees or buildings in path, plot of land cleared for $300 - 400$ m ahead, like farmlands.

Suburban areas: that are typically villages or highways scattered with trees and houses, containing some obstacles but not very congested.

Urban areas: which define built-up cities or large towns with large buildings and houses with two or more storeys, or larger villages with close houses and tall, thickly grown trees.

The median path loss is given by

$$\begin{array}{ll} \text{Suburban} & L_{50}\,(\text{dB}) = A + B\log_{10} R - C \\ \text{Open} & L_{50}\,(\text{dB}) = A + B\log_{10} R - D \\ \text{Urban} & L_{50}\,(\text{dB}) = A + B\log_{10} R - E \end{array} \tag{9.3.1}$$

where

$$A = 69.55 + 26.16\log_{10} f_c - 13.82\log_{10} h_b \tag{9.3.2a}$$

$$B = 44.9 - 6.55\log_{10} h_b \tag{9.3.2b}$$

$$C = 2\left[\log_{10}\left(\frac{f_c}{28}\right)\right]^2 + 5.4 \tag{9.3.2c}$$

$$D = 4.78\left(\log_{10} f_c\right)^2 + 18.33\log_{10} f_c + 40.94 \tag{9.3.2d}$$

$$E = 3.2\left[\log_{10}\left(11.75 h_m\right)\right]^2 - 4.97 \quad \text{for large cities and } f_c \geq 300\,\text{MHz} \tag{9.3.2e}$$

$$E = 8.29\left[\log_{10}\left(1.54 h_m\right)\right]^2 - 1.1 \quad \text{for large cities and } fc < 300\,\text{MHz} \tag{9.3.2f}$$

$$E = \left(1.1\log_{10} f_c - 0.7\right) h_m - \left(1.56\log_{10} f_c - 0.8\right) \quad \text{for medium to small cities} \tag{9.3.2g}$$

In these expressions, h_b is the base station antenna height in m, h_m is the mobile station antenna height in m, f_c is the communication frequency in MHz, R is the distance of the mobile to the base station in km. This model gives a path loss model with a range dependency R^{-n} where $n = B/10$. Typically this number is a little less than 4 and decreases as the base station height is increased. The model is valid only for $150\,\text{MHz} \leq f_c \leq 1500\,\text{MHz}$, $30\,\text{m} \leq h_b \leq 200\,\text{m}$, $1\,\text{m} < h_m < 10\,\text{m}$ and $R > 1\,\text{km}$.

9.3.2. COST-231 Hata Model. The Okumura–Hata model for medium to small cities has been extended to cover the band $1500\,\text{MHz} < f_c < 2000\,\text{MHz}$, [**64**]. The formula is

$$L_{50}\,(\text{dB}) = F + B\log_{10} R - E + G \tag{9.3.3}$$

where

$$F = 46.3 + 33.9\log_{10} f_c - 13.82\log_{10} h_b. \tag{9.3.4}$$

E and B are as defined in (9.3.2) for medium to small cities and

$$G = \begin{cases} 0 & \text{dB for medium-sized cities and suburban areas} \\ 3 & \text{dB for metropolitan areas} \end{cases} \tag{9.3.5}$$

9.3.3. Other Models. There are many other models for median path loss calculations. We will not discuss these models here since the idea is to give an understanding of the approach and not the details of the calculations. We will, however, give a small list of more commonly referred models in the literature. The details of these models can be found, for example, in [**65**] and [**66**].

Among these models we may mention the Walfish model, [**67**] which is a model that is based on theoretical calculations. A similar model is the Ikegami model, [**68**]. Later, based on extensive measurement campaigns in several European cities, COST-231 Action came up with an empirical model by combining the formulations of Walfish and Ikegami. The model, designated by COST-231 Walfish-Ikegami model, is applicable when the urban area has buildings of roughly the same heights and separation over the frequency range $800-2000\,\mathrm{MHz}$, base station height range of $4-50\,\mathrm{m}$, mobile station height range of $1-3\,\mathrm{m}$, and distance range of $20\,\mathrm{m}-5\,\mathrm{km}$. Models for calculation of rooftop diffraction, and flat edge model which takes the buildings as prismatic structures have also been proposed. The Longley-Rice model is a path loss (median) prediction model over irregular terrain suitable for coverage prediction in rural type of environments, and basically covers what has been discussed in the first six chapters.

9.4. Coverage Calculations

In broadcasting, base station applications and radar, an important thing is to determine the coverage area of the transmitter, especially in the planning of such systems. The coverage area is defined as that region for which the received signal strength exceeds a particular threshold for a particular percentage of time. For example, FCC provides $F(50,50)$ curves which gives the estimated field strength exceeded at 50% of the potential receiver locations for 50% of the time, at a receiving antenna height of $9\,\mathrm{m}$. Such curves can be used to obtain coverage diagrams.

In GSM application determination of the coverage area of a base station is quite important. Such a calculation is best explained by way of an example. Let us assume that the threshold signal level is x_0 (dBm) which is determined by the sensitivity of a typical GSM receiver. For given system parameters such as frequency, antenna heights, gains, and transmitter power, the median signal level can be written as

$$x_{50}(r)\ \mathrm{dB} = A(R) - 10n\log_{10}\frac{r}{R} \tag{9.4.1}$$

where R is a reference distance and $A(R)$ is the median signal level at $r=R$. This means that the median signal strength follows a r^{-n} distance dependence, where n is the path loss exponent. The term $A(R)$ depends on the system parameters. Let us also assume that the path loss (in dB) has a log-normal distribution. Then the signal strength is also log-normal and the signal P at a certain location can be written as

$$X = x_{50} + \sigma\tilde{x} \tag{9.4.2}$$

where $\tilde{x}$ is a zero mean, unit variance Gaussian random variable, and σ defines the standard deviation of the signal strength (or equivalently the path loss). The pdf of the signal is then

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-x_{50})^2}{2\sigma^2}\right]. \tag{9.4.3}$$

Therefore, at a location a distance r away from the transmitter, the probability that X exceeds the threshold level x_0 is given by

$$P(X > x_0) = P_{x_0}(r) = \int_{x_0}^{\infty} p(x)\,dx = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x_0 - x_{50}}{\sqrt{2}\sigma}\right) \tag{9.4.4}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du. \tag{9.4.5}$$

Since x_{50} is a function of r, this probability is also a function of r. Let us consider a circular area centered about the transmitter with radius R. The fraction of locations that the signal will exceed the threshold within this area is given by

$$A_c = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} P_{x_0}(r)\, r\,dr\,d\phi = \frac{2}{R^2}\int_0^R P_{x_0}(r)\, r\,dr \tag{9.4.6}$$

$$= \frac{1}{2} - \frac{1}{R^2}\int_0^R r\,\text{erf}\left(\frac{x_0 - x_{50}}{\sqrt{2}\sigma}\right) dr \tag{9.4.7}$$

Let us define $\alpha = (x_0 - A(R))/\sqrt{2}\sigma$ and $\beta = (10n\log_{10} e)/\sqrt{2}\sigma$. The integral can be evaluated using integration by parts and the result is

$$\frac{1}{R^2}\int_0^R r\,\text{erf}\left(\alpha + \beta\ln\frac{r}{R_0}\right) dr = \frac{1}{2}\text{erf}(\alpha) - \frac{\beta}{R^2\sqrt{\pi}}\int_0^R r\exp\left[-\left(\alpha + \beta\ln\frac{r}{R}\right)^2\right] dr \tag{9.4.8}$$

$$= \frac{1}{2}\text{erf}(\alpha) - \frac{1}{2}e^{-\frac{2\alpha\beta - 1}{\beta^2}}\left[1 - \text{erf}\left(\frac{1-\alpha\beta}{\beta}\right)\right] \tag{9.4.9}$$

Using this result in (9.4.7) gives

$$A_c = \frac{1}{2}\left[1 - \text{erf}(\alpha) + e^{-\frac{2\alpha\beta - 1}{\beta^2}}\left(1 - \text{erf}\left(\frac{1-\alpha\beta}{\beta}\right)\right)\right] \tag{9.4.10}$$

$$= P_{x_0}(R) + \frac{1}{2}e^{-\frac{2\alpha\beta - 1}{\beta^2}}\left[1 - \text{erf}\left(\frac{1-\alpha\beta}{\beta}\right)\right]. \tag{9.4.11}$$

As a numerical example, assume that $\sigma = 9$ dB, and $n = 3$. In (9.4.1) if we choose R as the distance for which $A(R) = x_{50} = x_0$, we have $\alpha = 0$. Then we get

$$A_c = P_{x_0}(R) + \frac{1}{2}e^{\frac{1}{\beta^2}}\left[1 - \text{erf}\left(\frac{1}{\beta}\right)\right] \tag{9.4.12}$$

$$= P_{x_0}(R) + K(\beta). \tag{9.4.13}$$

Note that $K(\beta)$ is always positive. This equation tells us if R is the distance at which $x_{50} = x_0$, i.e., half of the points on the boundary of the circle exceeds the threshold ($P_{x_0}(R) = 1/2$), then the fraction of points with signal strength exceeding the threshold inside the circle will be greater than 50%. For example if $\sigma = 9$ dB, and $n = 3$, we have $\beta = 1.024$ and $A_c = 0.72$. So, if half of the points on the circumference of a circle of radius R exceeds the threshold, 72% of the points inside the circle will exceed the threshold.

If we want to determine the radius for which a given percentage of the points inside the circle will exceed the threshold, we have to solve the problem backwards. Note that β depends only on the path loss model parameters σ and n. Once these values are given, we can calculate $K(\beta)$ in (9.4.12). Then, since A_c is specified, we can determine $P_{x_0}(R)$ from which, using (9.4.4), we can find $x_0 - x_{50}$. For example, let us again assume that $n = 3$ and $\sigma = 9$ dB. Furthermore, with the given frequency, antenna heights, gains, and transmitter power we calculate $A(5\,\text{km}) = -70$ dBm. This means that $x_{50} = -70$ dBm at a distance of 5 km. Suppose also that we want a 90% coverage, i.e., the threshold level must be exceeded at 90% of the locations. We have

$$K(\beta) = \frac{1}{2}e^{\frac{1}{\beta^2}}\left[1 - \text{erf}\left(\frac{1}{\beta}\right)\right] = 0.217$$

which gives

$$P_{x_0}(R) = 0.9 - 0.217 = 0.683\,.$$

Using (9.4.4) we find $\alpha = -0.337$ or $x_{50} = x_0 + 4.285$. The typical receiver sensitivity is -100 dBm, and we set x_0 to this level. Thus, we need $x_{50} = -95.715$ dBm. Using (9.4.1) we get

$$-95.715 = -70 - 30\log_{10}\frac{r}{1\,\mathrm{km}}$$

which gives $r = 7.2\,\mathrm{km}$.

9.5. Examples

EXAMPLE 38. *Considering a frequency of* $1.8\,\mathrm{GHz}$, $h_b = 20\,\mathrm{m}$, $h_m = 2\,\mathrm{m}$, *determine the path loss at a distance of* $R = 2\,\mathrm{km}$ *in suburban areas, using Okumura-Hata and COST-231 Hata models.*

SOLUTION 38. *The validity region of the Okumura-Hata model does not include* $1800\,\mathrm{MHz}$. *Still, with this model we would have*

$$\begin{aligned} A &= 69.55 + 26.16\log_{10} f_c - 13.82\log_{10} h_b = 136.7, \\ B &= 44.9 - 6.55\log_{10} h_b = 36.4, \\ C &= 2\left[\log_{10}\left(\frac{f_c}{28}\right)\right]^2 + 5.4 = 11.9. \end{aligned}$$

The median path loss is then given as

$$L_{50}(dB) = 136.7 + 36.4\log_{10} 2 - 11.9 = 135.8 \;\; dB.$$

For the COST-231 Hata model we have

$$\begin{aligned} B &= 44.9 - 6.55\log_{10} h_b = 36.4, \\ E &= 3.2\left[\log_{10}(11.75 h_m)\right]^2 - 4.97 = 6.0, \\ F &= 46.3 + 33.9\log_{10} f_c - 13.82\log_{10} h_b = 138.7, \\ G &= 0. \end{aligned}$$

The median path loss is then

$$L_{50}(dB) = 138.7 + 36.4\log_{10} 2 - 6 = 143.7 \;\; dB.$$

Note that the results of the two models differ by 7.9 dB. Since Okumura-Hata model is not valid for $1800\,\mathrm{MHz}$, *we should use the result of COST-231 Hata model.*

EXAMPLE 39. *Determine the median path loss at a frequency of* $900\,\mathrm{MHz}$ *for base station antenna height of* $h_b = 40\,\mathrm{m}$, *mobile station antenna height of* $h_m = 2\,\mathrm{m}$, *and a distance of* $R = 2\,\mathrm{km}$ *in a large city. How much the result would differ in a medium city?*

SOLUTION 39. *For these parameters we should use the Okumura-Hata model. We have*

$$\begin{aligned} A &= 69.55 + 26.16\log_{10} f_c - 13.82\log_{10} h_b = 124.7, \\ B &= 44.9 - 6.55\log_{10} h_b = 34.4, \\ E &= 3.2\left[\log_{10}(11.75 h_m)\right]^2 - 4.97 = 1.0 \quad \textit{for large cities}, \\ E &= (1.1\log_{10} f_c - 0.7)\, h_m - (1.56\log_{10} f_c - 0.8) = 1.3 \quad \textit{for medium to small cities}. \end{aligned}$$

Then the median path loss is

$$\begin{aligned} L_{50}\,(dB) &= 134.0 \quad \textit{for large cities,} \\ L_{50}\,(dB) &= 133.8 \quad \textit{for medium to small cities.} \end{aligned}$$

The difference is 0.25 *dB.*

EXAMPLE 40. *Suppose that the propagation in a certain region has* r^{-3} *dependence with a log-normal distribution with* $\sigma = 9$ *dB. For a certain system, the* 90% *coverage area has a radius of* 5 km. *If we increase the transmitter power by* 10 *dB, what will be the radius of the new* 90% *coverage area?*

SOLUTION 40. *From the given values we calculate* $\beta = 1.024$ *and* $K(\beta) = 0.217$. *Using (9.4.12) we find* $P_{x_0}(R) = A_c - K(\beta) = 0.9 - 0.217 = 0.683$. *Solving*

$$P_{x_0}(5\,\mathrm{km}) = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{x_0 - x_{50}}{\sqrt{2}\sigma}\right)$$

we find $x_0 - x_{50} = -1.428$ *dB. Increasing the transmitter power by* 10 *dB increases* x_{50} *by the same amount and the new value of* $x_0 - x_{50}$ *becomes* -11.428 *dB. Since we are again looking for* 90% *coverage radius, we must find the new distance* R_1 *for which* $P_{x_0}(R_1) = 0.683$ *or equivalently we must solve*

$$x_0 - x_{50}(R_1) = -1.428.$$

The propagation model is

$$x_{50}(r) = A(R) - 10n\log_{10}\frac{r}{R}.$$

Increasing the transmitter power increases $A(R)$ *by the same amount. That is we need to solve*

$$A(R) - 30\log_{10}\frac{5\,\mathrm{km}}{R} = A(R) + 10 - 30\log_{10}\frac{R_1}{R}.$$

Thus the new 90% *coverage radius will be* $R_1 = 10.8\,\mathrm{km}$. *Note that the previous calculations are not necessary and only given to explain that the problem is to find the new radius that will give the same value of* x_{50}. *In fact, the only parameters necessary to solve this problem is the attenuation model (i.e., the value of* n*) and the amount of increase in the transmitter power. Even the actual value of coverage percentage is not necessary, since the coverage percentage is required to be same in the two cases.*

EXAMPLE 41. *With the parameters of Example 40, how much the transmitter power should be increased to ensure a* 99.9% *coverage for the same area of radius* 5 km*?*

SOLUTION 41. *From calculations of Example 40, we find* P_{x_0} *for* 99.9% *coverage to be*

$$P_{x_0}(R) = A_c - K(\beta) = 0.999 - 0.217 = 0.782.$$

Solving

$$P_{x_0}(5\,\mathrm{km}) = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{x_0 - x_{50}}{\sqrt{2}\sigma}\right)$$

we determine $x_0 - x_{50} = -2.337$ *is required. Thus, we need to increase* x_{50} *by* $2.337 - 1.428 = 0.909$ *dB, which is the required increase in the transmitter power.*