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CHAPTER 8

Numerical Methods for Propagation Calculations

The basic problem in propagation calculations is to solve the Maxwell's equations with some specified boundary and initial conditions. Basically the problem can be formulated as an integral equation which can be solved by an appropriate numerical method. Although such a solution is possible in principle, the complexity of the problem is overwhelming. The main difficulty will be an accurate description of the radio environment which must have a subwavelength accuracy. The second difficulty will be the size of the problem. Assume that we want to determine the coverage of a GSM900 base station serving a macrocell of radius 1 km. We may take the computational domain as a cylinder of height about 100 m which will ensure that the upper boundary of the computational domain is sufficiently far away from any scattering object. If we discretize the volume using cubical grids, the size of the grid should not exceed a fraction of the wavelength. Let us take the grid size as $\lambda/2$, which is the largest grid size that can be tolerated in a numerical solution. The number of grid points will then be approximately $\pi r^2 h/(\lambda/2)^3 \approx 6.8 \times 10^{10}$. At each grid point we have three unknowns corresponding to the components of the electric field which will give 2.0×10^{11} unknowns. With a straightforward numerical method this would require the inversion of 2.0×10^{11} by 2.0×10^{11} complex matrix. If we use double precision arithmetic, each entry of this matrix will be represented by 16 bytes. The memory required to store this matrix is 6.4×10^{23} bytes or 6.4×10^{11} Tbytes, a number that exceeds the total storage capacity of all storage devices available on earth!

These calculations show that a direct solution of the vector Maxwell's equations for propagation prediction is practically impossible. The approach is then to use certain simplifying assumptions which will render the problem tractable. Many different approximations have been proposed in the literature. In this chapter, we will discuss only a few approaches that are more commonly considered in the literature.

8.1. Integral Equation Methods

Many different integral equation formulations are possible for the propagation problem, including EFIE, MFIE or mixed formulations. Here we will consider one derivation that uses the Green's integral theorem, [**54**]. We will consider only an elevated vertical electrical dipole over an irregular terrain. Let the vertical component of the electric field be denoted by ψ. Then, assuming a homogeneous atmosphere with unity index of refraction, the field ψ must satisfy the scalar Helmholtz equation

$$\nabla^2\psi + k_0^2\psi = -I\delta(\mathbf{r}) \tag{8.1.1}$$

where I denotes the amplitude of the source. We have chosen a spherical coordinate system with origin at the transmitting antenna. At a point P on the surface of the ground, the field

ψ must satisfy the Leontovitch boundary condition

$$\left.\frac{\partial\psi}{\partial n}\right|_P = jk_0\psi\left(P\right)\Delta \tag{8.1.2}$$

where $\Delta = \sqrt{(\kappa-1)}/\kappa$ is the normalized surface impedance at point P. Let $G\left(\mathbf{r};\mathbf{r}'\right)$ be the free space scalar Green's function, i.e.,

$$G\left(\mathbf{r};\mathbf{r}'\right) = \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}. \tag{8.1.3}$$

Using Green's second identity we can write

$$\int_V \left(\psi\nabla^2 G - G\nabla^2\psi\right) dV' = \oint_S \left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n}\right) dS' \tag{8.1.4}$$

where $\mathbf{n}$ denotes and outward normal to the surface S bounding the volume V. We choose the volume V as a hemispherical volume bounded by the ground surface below and a hemispherical surface above, excluding a small spherical volume centered the receiver at $\mathbf{r}$. Let $\mathbf{r}'$ denote a point P on the ground.

We can consider the surface S as the sum of three surfaces: The ground surface S_G, the small spherical surface S_0 about the receiver, the upper hemisphere S_∞ whose radius will be taken as infinitely large. The surface integral over S_∞ vanishes due to radiation condition, and the integral over S_0 is simply $\psi\left(\mathbf{r}\right)$. Thus, the surface integral on the right hand side of (8.1.4) becomes

$$\begin{aligned}\oint_S \left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n}\right) dS' &= \psi\left(\mathbf{r}\right) + \int_{S_G}\left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n}\right) dS', \\ &= \psi\left(\mathbf{r}\right) + \int_{S_G}\left(jk_0\Delta\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}\psi\left(\mathbf{r}'\right) - \psi\left(\mathbf{r}'\right)\frac{\partial}{\partial n}\left(\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}\right)\right) dS'. \end{aligned} \tag{8.1.5}$$

The normal derivative of the Green's function can be written as

$$\frac{\partial}{\partial n}\left(\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}\right) = \frac{\partial}{\partial\left|\mathbf{r}-\mathbf{r}'\right|}\left(\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}\right)\frac{\partial\left|\mathbf{r}-\mathbf{r}'\right|}{\partial n}, \tag{8.1.6}$$

$$= -jk_0\left(1+\frac{1}{jk_0 r}\right)\frac{e^{-jk_0 r}}{4\pi r}\frac{\partial\left|\mathbf{r}-\mathbf{r}'\right|}{\partial n}, \tag{8.1.7}$$

and the surface integral becomes

$$\begin{aligned}&\oint_S \left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n}\right) dS' = \\ &\quad \psi\left(\mathbf{r}\right) + \int_{S_G} jk_0\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}\psi\left(\mathbf{r}'\right)\left[\Delta + \left(1+\frac{1}{jk_0\left|\mathbf{r}-\mathbf{r}'\right|}\right)\frac{\partial\left|\mathbf{r}-\mathbf{r}'\right|}{\partial n}\right] dS'.\end{aligned} \tag{8.1.8}$$

Next we consider the volume integral on the left hand side of (8.1.4). Since the Green's function must satisfy the Helmholtz equation, we have

$$\nabla^2 G\left(\mathbf{r};\mathbf{r}'\right) + k_0^2 G\left(\mathbf{r};\mathbf{r}'\right) = -\delta\left(\mathbf{r}-\mathbf{r}'\right) \tag{8.1.9}$$

which gives $\nabla^2 G(\mathbf{r};\mathbf{r}') = -k_0^2 G(\mathbf{r};\mathbf{r}') - I\delta(\mathbf{r}-\mathbf{r}')$. But since the point $\mathbf{r}'$ is excluded from the volume we can omit the Dirac delta term and write $\nabla^2 G(\mathbf{r};\mathbf{r}') = -k_0^2 G(\mathbf{r};\mathbf{r}')$ inside V. Similarly, $\nabla^2\psi$ can be written from (8.1.1) as

$$\nabla^2\psi(\mathbf{r}) = -k_0^2\psi(\mathbf{r}) - I\delta(\mathbf{r}). \tag{8.1.10}$$

The volume integral can now be written as

$$\int_V \left(\psi\nabla^2 G - G\nabla^2\psi\right) dV' = \int_V G(\mathbf{r};\mathbf{r}')\,\delta(\mathbf{r})\,dV' \tag{8.1.11}$$

$$= \psi_{\mathbf{0}}(\mathbf{r}) \tag{8.1.12}$$

where $\psi_{\mathbf{0}}(\mathbf{r})$ denotes the free space field at the receiver location due to the transmitting antenna which can be written as

$$\psi_{\mathbf{0}}(\mathbf{r}) = f(\mathbf{r})\frac{e^{-jk_0 r}}{4\pi r} \tag{8.1.13}$$

where $f(\mathbf{r})$ accounts for the transmitting antenna pattern, and $r = |\mathbf{r}|$. The attenuation factor is related to the actual field at the receiver, $\psi(\mathbf{r})$ by the relation

$$\psi(\mathbf{r}) = F(\mathbf{r})\frac{e^{-jk_0 r}}{4\pi r}. \tag{8.1.14}$$

With these expressions, (8.1.4) becomes

$$\psi_{\mathbf{0}}(\mathbf{r}) = \psi(\mathbf{r}) + \int_{S_G} jk_0 \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\psi(\mathbf{r}')\left[\Delta + \left(1+\frac{1}{jk_0|\mathbf{r}-\mathbf{r}'|}\right)\frac{\partial|\mathbf{r}-\mathbf{r}'|}{\partial n}\right] dS'. \tag{8.1.15}$$

Using (8.1.13) and (8.1.14) in (8.1.15) yields

$$F(\mathbf{r}) = f(\mathbf{r}) - \frac{jk_0}{4\pi}\int_{S_G} F(\mathbf{r}')\exp\left[-jk_0\left(|\mathbf{r}-\mathbf{r}'| + r' - r\right)\right] \times \frac{r}{|\mathbf{r}-\mathbf{r}'|\,r'}\left[\Delta + \left(1+\frac{1}{jk_0|\mathbf{r}-\mathbf{r}'|}\right)\frac{\partial|\mathbf{r}-\mathbf{r}'|}{\partial n}\right] dS'. \tag{8.1.16}$$

This equation is the desired integral equation in the unknown attenuation function $F(\mathbf{r})$. However, the integration over S_G is a double integral which increases the size of the problem. Furthermore, the integrand is very oscillatory due to the $\exp\left[-jk_0\left(|\mathbf{r}-\mathbf{r}'| + r' - r\right)\right]$ term. The oscillatory term suggests that a stationary phase approximation is suitable. Projecting the integration over the earth's surface onto a horizontal plane and introducing elliptical coordinates with foci at the transmitter and the receiver, the method of stationary phase enables us to reduce the two-dimensional to a one dimensional integral along the line joining the transmitter to the receiver. Details of this procedure is explained in [**54**]. Since the distances of interest are typically much larger than the wavelength, the term $1/jk_0|\mathbf{r}-\mathbf{r}'|$

can also be neglected. Finally, the integral equation becomes

$$F(x) = f(x) - \sqrt{\frac{jx}{\lambda}} \int_0^x F(u) \exp\left[-jk_0\left(|\mathbf{r} - \mathbf{r}'| + r' - r\right)\right] \times \left(\Delta(u) + \frac{d|\mathbf{r} - \mathbf{r}'|}{dn}\right) \frac{du}{\sqrt{u(x-u)}}. \quad (8.1.17)$$

A numerical solution can be carried out by using the method of moments.

8.2. Parabolic Equation

The wave equation is a hyperbolic equation which is generally hard to solve by numerical techniques. The parabolic equation is an approximation to the wave equation which uses a preferred direction of motion and models the energy propagated in a cone centered about the propagation direction, which is called the *paraxial direction*. It was first introduced in the 1940's, [**55**], to solve the problem of radio wave propagation around the earth.

8.2.1. Derivation of the Parabolic Equation. Consider a plane wave propagating in the z-direction which is chosen to be along the surface of the earth. We will use a cylindrical coordinate system (z, ϕ, x) where z is the distance from the cylindrical axis. Let ψ be an electromagnetic field component (E_ϕ or H_ϕ) which must satisfy the scalar wave equation in three dimensions, i.e.,

$$\nabla^2\psi + k_0^2 n^2 \psi = 0 \quad (8.2.1)$$

where k_0 is the wavenumber in free space and n is the refractive index of the medium which is a function of position in general. We first use the earth flattening formulation which allows us to represent the wave equation in cylindrical coordinates with a modified refractive index

$$m(x, z, \phi) = n(x, z, \phi) + \frac{z}{a}. \quad (8.2.2)$$

The wave equation in cylindrical coordinates is then

$$\frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial\psi}{\partial z}\right) + \frac{1}{z^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial x^2} + k_0^2 m^2(x, z)\psi = 0. \quad (8.2.3)$$

If we assume that the field is independent of the angle ϕ (which is generally a good assumption if the transmitting antenna has a pattern that is symmetrical about its main beam) we can write

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{1}{x}\frac{\partial\psi}{\partial x} + k_0^2 m^2(x, z)\psi = 0. \quad (8.2.4)$$

The unusual choice of coordinate variables is now apparent. The x variable denotes the range and the z variable is height above the earth's surface.

The wave equation in free space has the outgoing solution $H_0^{(2)}(kx)$ which has an asymptotic expansion

$$H_0^{(2)}(kx) \sim \sqrt{\frac{2}{\pi kx}} e^{j\pi/4} e^{-jkx}. \quad (8.2.5)$$

Therefore we may assume a solution of the form

$$\psi(x,z) = \frac{e^{-jk_0 x}}{\sqrt{x}} u(x,z). \tag{8.2.6}$$

The rapid phase variation and the expected $1/\sqrt{x}$ of $\psi(x,z)$ are thus separated from the function $u(x,z)$ and therefore it varies slowly in the propagation direction x.

Using (8.2.6) in (8.2.4) yields

$$\frac{\partial^2 u(x,z)}{\partial z^2} + \frac{\partial^2 u(x,z)}{\partial x^2} - 2jk_0 \frac{\partial u(x,z)}{\partial x} + k_0^2 \left(m^2 - 1 + \frac{1}{(2k_0 x)^2} \right) u(x,z) = 0. \tag{8.2.7}$$

As far as we are interested in the fields at points far away from the transmitter, we have $k_0 x \gg 1$ and we can drop the $1/(2k_0 x)^2$ term. Furthermore, the slow variation of the function $u(x,z)$ implies

$$\left| \frac{\partial^2 u(x,z)}{\partial x^2} \right| \ll 2k_0 \left| \frac{\partial u(x,z)}{\partial x} \right| \tag{8.2.8}$$

and we can drop the second derivative in x. These assumptions reduce (8.2.7) to

$$\frac{\partial^2 u(x,z)}{\partial z^2} - 2jk_0 \frac{\partial u(x,z)}{\partial x} + k_0^2 \left(m^2 - 1 \right) u(x,z) = 0 \tag{8.2.9}$$

which is a parabolic equation since only the first derivative in x appears. This final equation is known as the *parabolic equation* (PE) for electromagnetic waves.

The two basic assumptions we have made in deriving the PE are

(1) The field point is many wavelengths away from the source,
(2) The change in $\partial u/\partial x$ is small over a wavelength.

The first condition is not a restriction for propagation calculations. The second condition is generally satisfied for realistic refractive index profiles and within 15 to 20° above horizontal. Modification to the PE to allow the high angle limit is possible, but is much more complex and will be omitted in this discussion.

8.2.2. Solutions of the PE. The numerical solution of the PE is much easier than the original Maxwell's equations, which is hyperbolic in nature. The solution of the exact equation requires the specification of the boundary conditions on the boundaries of the two dimensional computational region. The parabolic equation on the other hand is an open boundary problem which can be solved by a "marching" technique. That is, the field variation on at a point $x + \Delta x$ can be determined from a knowledge of the field variation on a constant x surface.

8.2.2.1. *Finite Difference Solution.* Let us assume that the computational domain is discretized as shown in Fig. 8.1. The discretization steps in x and z directions are Δx and Δz, respectively. We use the notation

$$u_{p,q} = u(p\Delta x, q\Delta z)$$

where p and q are integers. Using central difference formula for the second derivative in z and forward difference formula for the derivative in x we get

$$\frac{u(x,z+\Delta z)-2u(x,z)+u(x,z-\Delta z)}{(\Delta z)^2} - 2jk_0\frac{u(x+\Delta x,z)-u(x,z)}{\Delta x}+k_0^2\left(m^2(x,z)-1\right)u(x,z)=0, \quad (8.2.10)$$

or

$$\frac{u_{p,q+1}-2u_{p,q}+u_{p,q-1}}{(\Delta z)^2}-2jk_0\frac{u_{p+1,q}-u_{p,q}}{\Delta x}+k_0^2\left(m_{p,q}^2-1\right)u_{p,q}=0$$

which can be solved for $u_{p+1,q}$ as

$$u_{p+1,q}=u_{p,q}+\frac{\Delta x}{2jk_0(\Delta z)^2}\left(u_{p,q+1}-2u_{p,q}+u_{p,q-1}\right)+\frac{(\Delta x)\,k_0^2\left(m_{p,q}^2-1\right)}{2jk_0}u_{p,q}. \quad (8.2.11)$$

Note that this expression is fully explicit, meaning that the values of $u(x+\Delta x,z)$ are expressed in terms of $u(x,z)$. We only need to know the value of $u(x,z)$ at $x=0$ which is an initial condition, and at $z=0$ and $z=z_{\max}$ which are boundary conditions on the earth's surface and at the upper boundary of the computational domain. If we try to solve a hyperbolic differential equation, we also need the boundary condition at $x=x_{\max}$, i.e., at the right end of the computational domain.

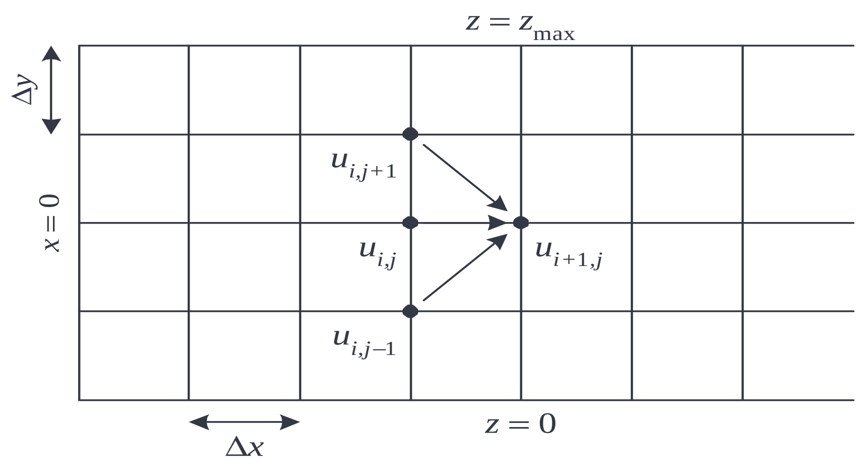

FIGURE 8.1. Solution domain for a two dimensional computational region.

8.2.2.2. *Split Step Algorithm.* The parabolic equation (8.2.9) can be written as

$$\frac{\partial u(x,z)}{\partial x}=-j\left(\frac{1}{2k_0}\frac{\partial^2}{\partial z^2}+\frac{k_0}{2}\left(m^2(x,z)-1\right)\right)u(x,z). \quad (8.2.12)$$

To explain the split step algorithm, let us first assume that the refractive index is a constant. Defining the operators

$$\mathcal{A} = \frac{1}{2k_0}\frac{\partial^2}{\partial z^2}, \tag{8.2.13}$$

$$\mathcal{B} = \frac{k_0}{2}\left(m^2 - 1\right) \tag{8.2.14}$$

we can write (8.2.12) formally as

$$\frac{\partial u\,(x, z)}{\partial x} = -j\,(\mathcal{A} + \mathcal{B})\,u\,(x, z) \tag{8.2.15}$$

and integrate over x to obtain

$$u\,(x + \Delta x, z) = e^{-j\Delta x(\mathcal{A}+\mathcal{B})}u\,(x, z)\,. \tag{8.2.16}$$

This result express the field at $x + \Delta x$ in terms of the field at x. We now want to split the exponential operator. The simplest split is

$$S = e^{-j\Delta x\mathcal{B}}e^{-j\Delta x\mathcal{A}}. \tag{8.2.17}$$

Such a splitting is possible only if the operators $\mathcal{A}$ and $\mathcal{B}$ commute. In the general case of variable refractive index this is not true since

$$\mathcal{A}\mathcal{B}u = \frac{1}{4}\frac{\partial^2}{\partial z^2}\left(m^2 - 1\right)u \neq \frac{1}{4}\left(m^2 - 1\right)\frac{\partial^2 u}{\partial z^2} = \mathcal{B}\mathcal{A}u. \tag{8.2.18}$$

However, when m is constant the operators commute. Taking the Fourier transform of (8.2.16) in the z variable and defining the Fourier variable in z as p we get

$$U\,(x + \Delta x, p) = e^{-j\Delta x\mathcal{B}}e^{-j\Delta x\mathcal{A}}U\,(x, p)\,. \tag{8.2.19}$$

Then the inverse Fourier transform gives

$$u\,(x + \Delta x, z) = \exp\left(\frac{-jk_0\Delta x\,(m^2 - 1)}{2}\right)F^{-1}\left\{\exp\left(\frac{j\Delta x p^2}{2k_0}\right)F\left\{u\,(x, z)\right\}\right\} \tag{8.2.20}$$

which marches the solution forward from x to $x + \Delta x$.

Let us define the commutator

$$[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \tag{8.2.21}$$

From the definition of the operators we have

$$[\mathcal{A}, \mathcal{B}]\,u = \frac{1}{4}\frac{\partial^2\,(m^2 - 1)\,u}{\partial z^2} - \frac{1}{4}\left(m^2 - 1\right)\frac{\partial^2 u}{\partial z^2} \tag{8.2.22}$$

$$= \frac{1}{4}u\frac{\partial^2 m^2}{\partial z^2} + \frac{1}{2}\frac{\partial m^2}{\partial z}\frac{\partial u}{\partial z}. \tag{8.2.23}$$

Thus, if the variations of the refractive index with height are relatively slow, the error incurred by splitting the exponential remains small.

The error introduced due to variations of refractive index with range can be understood by considering

$$E = e^{-j\Delta x\mathcal{B}}e^{-j\Delta x\mathcal{A}}u - e^{-j\Delta x(\mathcal{A}+\mathcal{B})}u. \tag{8.2.24}$$

Expanding the exponentials into Taylor series gives

$$E = \frac{1}{2}\Delta x^2 \left[\mathcal{A}, \mathcal{B}\right] u + hot \tag{8.2.25}$$

which shows that the error is of second order in the range step. These discussions show that if the index of refraction varies slowly with height and range (which is usually the case in the atmosphere) we can use (8.2.20) with a high degree of accuracy. The error in these approximations can be shown to be, [**56**],

$$\Delta x \left[-jk_0 mu \frac{\partial m}{\partial x} + m \frac{\partial m}{\partial z}\frac{\partial u}{\partial z} + \frac{mu}{2}\frac{\partial^2 m}{\partial z^2} + \frac{u}{2}\left(\frac{\partial m}{\partial z}\right)^2 \right] + O\left[(\Delta x)^2\right]. \tag{8.2.26}$$

This result shows that the error depends on the step size Δx, frequency and the variation of the refractive index. By decreasing the step size, the error can be made as small as desired.

The algorithm described by (8.2.20) is known as the split-step algorithm. In numerical implementation, the Fourier transforms are evaluated as discrete Fourier sums and implemented using the FFT algorithm. In approximating the Fourier transform by a DFT, care must be exercised to prevent aliasing. For a given FFT size, there is a maximum height in z and $k_0 \sin\theta$ (p) that can be represented without aliasing. There is a limit to the maximum angle of propagation which will be smaller for higher frequencies. This limit is not fundamental and can be overcome by using different splitting of the exponential operator. Wide angle PE algorithms have been proposed in the literature, [**57, 58**].

We have discussed a 2-D PE algorithm which assumes symmetry in the azimuthal direction. In practical problems, the terrain has variation in the transverse direction. A 3-D version of PE is also possible, [**59**], but will not be discussed here.

8.2.3. Initial Field and the Boundary Conditions. For the finite difference and the split-step Fourier algorithms, one needs the initial field $u(0, z)$ to start the solution procedure. The $x = 0$ boundary is chosen sufficiently far away from the transmitter and the initial field is taken as the far field antenna pattern. Typically a Gaussian pattern is chosen as a simplifying approximation. However, any pattern can be defined either analytically or numerically, provided that the beam width satisfies the maximum angle constraint for the frequency and FFT size chosen.

To propagate the solution in the x direction, one also needs the boundary conditions at the upper $(z = z_{\max})$ and the lower $(z = 0)$ surfaces. At the upper boundary, the field should not be reflected back into the computational domain and energy should not leak from the upper boundary to the lower boundary due to the circular nature of the FFT. One way to implement this is to chose the upper boundary sufficiently high so that the fields arriving at this surface is negligible. Another, and numerically more efficient way is to use an absorbing boundary condition at the upper boundary. Applying a windowed FFT algorithm to the data in the z direction accomplishes the same task.

If propagation over a plane earth is considered, the boundary condition at the lower boundary is simply defined by the electrical properties of the ground. This applies to mixed smooth earth problems. The boundary condition can also be applied by using image theory. The computational domain is doubled in the z direction about the $z = 0$ plane and the computations are carried out from $-z_{\max}$ to z. For each component in the plane wave spectrum of the source, an image is formed by multiplying the source amplitude by the

Fresnel reflection coefficient corresponding to its angle and the electrical parameters of the ground. Surface roughness can be taken into account in the calculation of the reflection coefficient. The total field due to source and image are then propagated in the x direction. This would be true if $|F| \approx 1$, since adding the two fields ignores the energy leakage into ground.

When the terrain is not flat, the finite difference solution is applicable in a straightforward manner. However, the split-step FFT algorithm is computationally much more efficient. The basic idea in applying split-step FFT for irregular terrains is to apply a conformal mapping that transforms the irregular terrain into a flat one. This approach is first presented in [**60**], and applied to tropospheric propagation in [**61**].

We define a new variable as

$$\zeta = z - t(x) \tag{8.2.27}$$

where $t(x)$ describes the actual terrain above the earth's surface. We then define the function $u(x, z)$ in terms of this new variable as

$$u(x, z) = U(x, \zeta)\, e^{-j\theta(x,\zeta)}. \tag{8.2.28}$$

Substituting (8.2.28) into (8.2.9) and applying the chain rule yields

$$\begin{aligned}\frac{\partial^2 U}{\partial \zeta^2} - 2j\left(\frac{\partial \theta}{\partial \zeta} + k_0 \frac{\partial \zeta}{\partial x}\right)\frac{\partial U}{\partial \zeta} - 2jk_0 \frac{\partial U}{\partial x} \\ + \left[k_0^2\left(m^2 - 1\right) - 2k_0 \frac{\partial \theta}{\partial x} - 2k_0 \frac{\partial \theta}{\partial \zeta}\frac{\partial \zeta}{\partial x} - j\frac{\partial^2 \theta}{\partial \zeta^2} - \left(\frac{\partial \theta}{\partial \zeta}\right)^2\right] U = 0\end{aligned} \tag{8.2.29}$$

where we have omitted the arguments of the functions for brevity. If this equation can be put into the same form as (8.2.9), the split-step FFT algorithm can be applied. To this end we impose the condition

$$\frac{\partial \theta}{\partial \zeta} + k_0 \frac{\partial \zeta}{\partial x} = 0 \tag{8.2.30}$$

which restricts the function θ to have a certain form. Since

$$\frac{\partial \zeta}{\partial x} = -t'(x) \tag{8.2.31}$$

we can write

$$\frac{\partial \theta}{\partial \zeta} = k_0 t'(x). \tag{8.2.32}$$

Therefor, $\theta(x, \zeta)$ must have the form

$$\theta(x, \zeta) = \zeta k_0 t'(x) + g(x) \tag{8.2.33}$$

where $g(\zeta)$ is an arbitrary function as yet. We can now write (8.2.29) as

$$\frac{\partial^2 U}{\partial \zeta^2} - 2jk_0 \frac{\partial U}{\partial x} + \left[k_0^2\left(m^2 - 1\right) - 2k_0^2 \zeta t''(x) - 2k_0 \frac{\partial g(x)}{\partial x} + \left(k_0 t'(x)\right)^2\right] U = 0. \tag{8.2.34}$$

A further simplification is achieved if we choose

$$\frac{\partial g(x)}{\partial x} = \frac{k_0}{2}\left(t'(x)\right)^2$$

and finally we can recast (8.2.34) as

$$\frac{\partial^2 U}{\partial \zeta^2} - 2jk_0\frac{\partial U}{\partial x} + k_0^2\left[\left(m^2-1\right) - 2\zeta t''(x)\right]U = 0. \tag{8.2.35}$$

This equation has the same form as (8.2.9) and the split-step algorithm can be applied to this equation. The solution will now become

$$U(x+\Delta x,\zeta) = \exp\left(\frac{-jk_0\Delta x\left(m^2\left(\zeta + t(x)\right) - 1 - 2\zeta t''(x)\right)}{2}\right)$$
$$F^{-1}\left\{\exp\left(\frac{j\Delta x p^2}{2k_0}\right)F\left\{U(x,\zeta)\right\}\right\}. \tag{8.2.36}$$

Obviously, m must be expressed in the new coordinates.

8.2.4. Some Result. In this section we will discuss some results obtained by using the PE equation, [**4**]. Fig. 8.2 shows the solution for a smooth spherical earth. The upward bending of the rays are due to mapping the earth surface on a plane. The bottom graph shows the field intensity at a constant height as a function of range. The interference pattern can easily be observed.

Figure 8.3 shows the propagation over a pyramidal hill. Edge diffraction effect can be observed.

As a last example, propagation over mixed flat earth is shown in Fig. 8.4. The geometry models a smooth region characterizing a sea-land-sea path. The bottom graph shows the field intensity at a constant height as a function of range. The recovery effect can be observed.

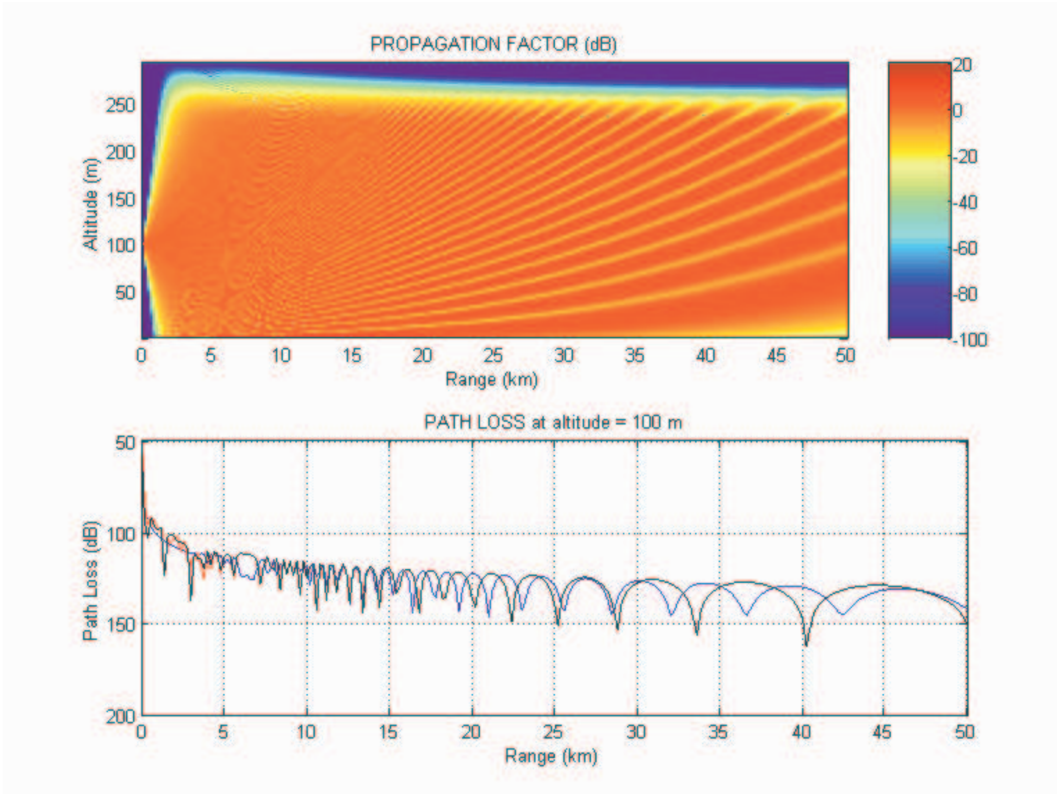

FIGURE 8.2. PE solution over a smmoth spherical earth with constant refractivity, [**4**].

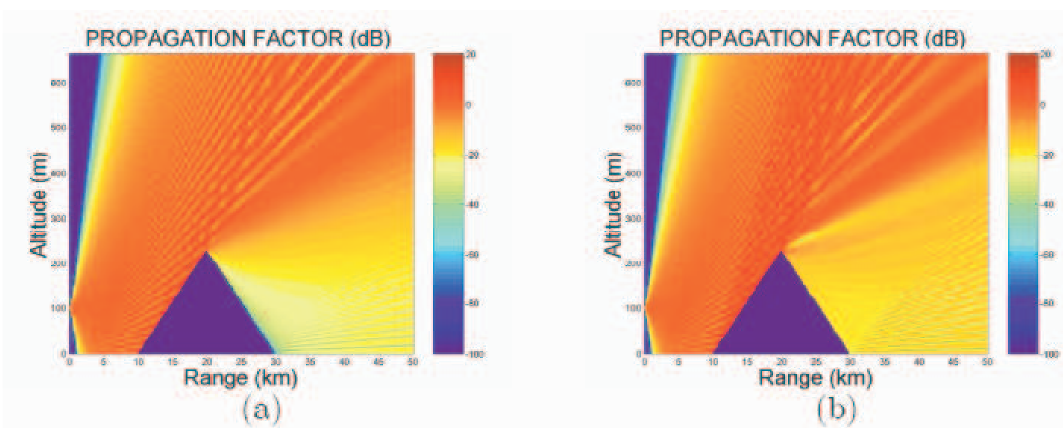

FIGURE 8.3. Pyramidal hill with (a) staircase approximation and (b) conformal mapping, [**4**]

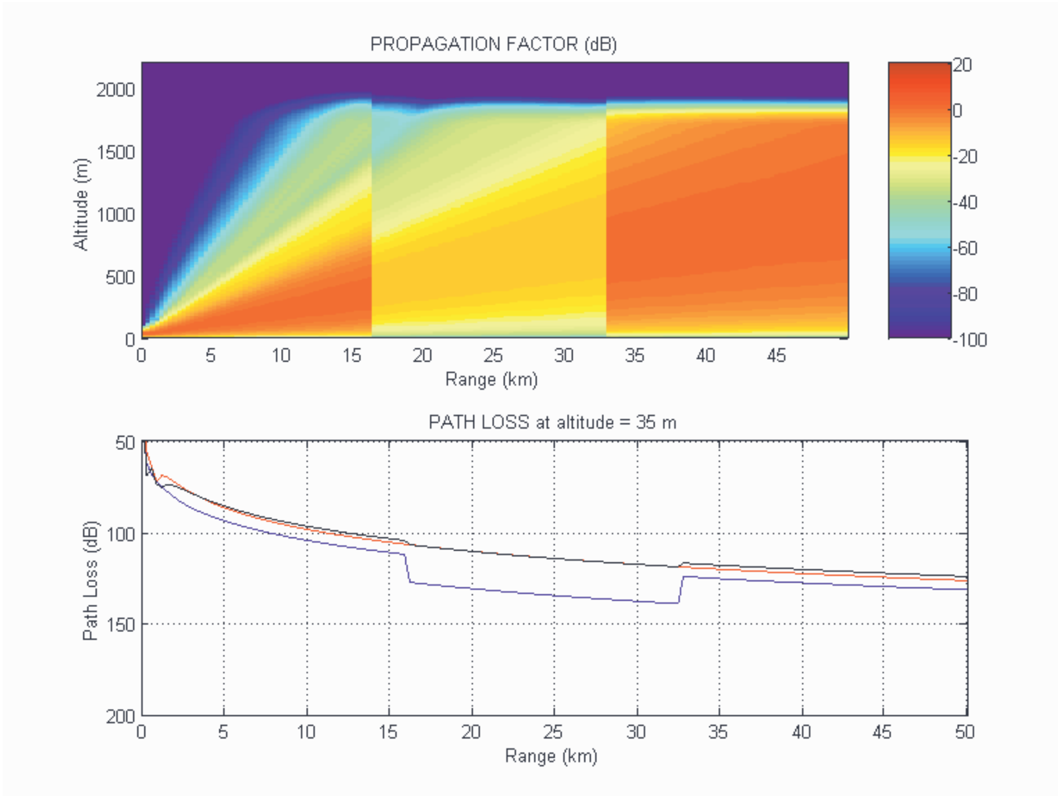

FIGURE 8.4. Propagation over mixed flat earth, [**4**].