Last revised April 11, 2011

CHAPTER 4

Propagation over Earth (Diffraction Region)

The rigorous solution of the propagation of radio waves over earth problem requires the solution of Maxwell's equations with appropriate boundary conditions. The optical approach discussed in the previous chapter is only approximate. It gives quite accurate results in the so called *interference region* which extends just short of the horizon. Especially in the case of spherical earth, the field beyond optical horizon is non-zero, which cannot be predicted by the ray theory. Thus we need a better formulation that gives the fields in this so called *diffraction region*, beyond horizon. We will consider the propagation over a flat earth first.

4.1. Current Element over a Flat Surface

4.1.1. Vertical Current Element. The problem of radiation from a vertical current element over an infinite flat lossy dielectric medium was first solve by Sommerfeld, [**23**]. The solution was later expressed in an approximate form by Norton, [**24**], that is much more useful in engineering applications. The following derivation is based on the derivation by Collin, [**25**].

Consider a z-directed current element of unit strength located at a height h above a flat dielectric surface characterized by the complex dielectric constant

$$\kappa = \kappa' - j\kappa'' = \kappa' - j\frac{\sigma}{\omega\epsilon_0} \tag{4.1.1}$$

where σ is the conductivity, κ' is the dielectric constant (relative permittivity). The pertinent geometry is shown in Fig. 4.1. Since the current element is assumed to be in the z-direction, the vector potential has only the z component and must satisfy

$$\begin{aligned} \nabla^2 A_z + k_0^2 A_z &= -\mu_0 \delta(x)\,\delta(y)\,\delta(z-h)\,, \quad & z>0, \\ \nabla^2 A_{z3} + k^2 A_{z3} &= 0, & z<0. \end{aligned} \tag{4.1.2a}$$

The Fourier transforms of these equation with respect to x and y variables are

$$\left(\frac{\partial^2}{\partial z^2} + k_0^2 - \beta^2\right)\mathcal{A}_z\left(\beta_x, \beta_y, z\right) = -\mu_0 \delta(z-h)\,, \qquad z>0, \tag{4.1.3a}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2 - \beta^2\right)\mathcal{A}_{z3}\left(\beta_x, \beta_y, z\right) = 0, \qquad z<0. \tag{4.1.3b}$$

where $\beta = \beta_x^2 + \beta_y^2$, and

$$\mathcal{A}_z\left(\beta_x, \beta_y, z\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_z(x,y,z)\, e^{j\beta_x x} e^{j\beta_y y} dxdy, \tag{4.1.4a}$$

$$\mathcal{A}_{z3}\left(\beta_x, \beta_y, z\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{z3}(x,y,z)\, e^{j\beta_x x} e^{j\beta_y y} dxdy. \tag{4.1.4b}$$

The continuity of the tangential components of the electric and magnetic field at $z = 0$ requires

$$\mathcal{A}_z\left(\beta_x, \beta_y, 0^+\right) = \mathcal{A}_{z3}\left(\beta_x, \beta_y, 0^-\right), \tag{4.1.5a}$$

$$\left.\frac{\partial \mathcal{A}_z\left(\beta_x, \beta_y, z\right)}{\partial z}\right|_{z=0^+} = \frac{1}{\kappa}\left.\frac{\partial \mathcal{A}_{z3}\left(\beta_x, \beta_y, 0\right)}{\partial z}\right|_{z=0^-}. \tag{4.1.5b}$$

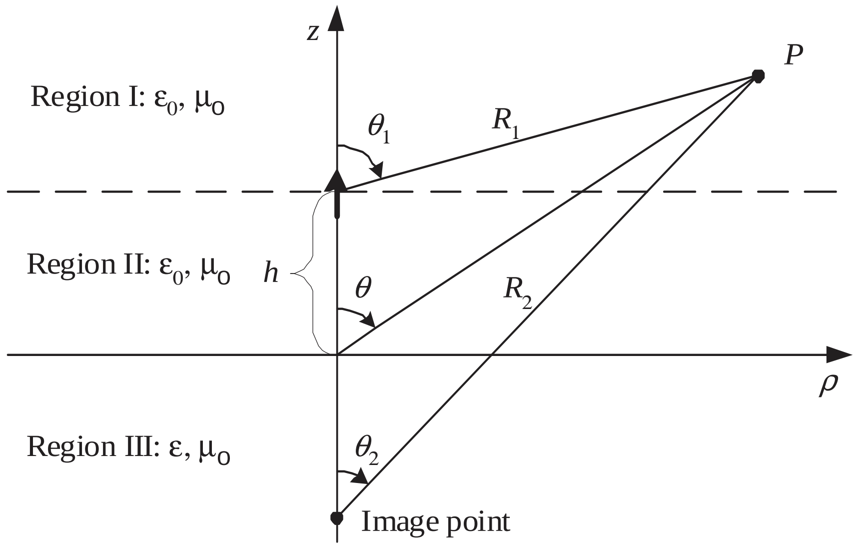

FIGURE 4.1. The geometry for the solution of Sommerfeld problem.

We divide the $z > 0$ region into two regions: $z > h$ is the Region I and $0 < z < h$ is the Region II. Since the differential equations in (4.1.3) are harmonic equations in z, their solutions can be expressed as linear combinations of the harmonic functions $e^{\pm j\gamma_0 z}$ where $\gamma_0 = \sqrt{k_0^2 - \beta^2}$ with negative imaginary part. In Region I, the solution must decay as $z \to \infty$ hence only $e^{-j\gamma_0 z}$ term can be used. In Region II both solutions are possible, and a linear combination of $e^{-j\gamma_0 z}$ and $e^{+j\gamma_0 z}$ terms must be used. For $z < 0$, only a downward decaying wave is admissible. Furthermore, the function $\mathcal{A}_z$ must be continuous at $z = 0$ and $z = h$. Thus we can write

$$\mathcal{A}_{z1}\left(\beta_x, \beta_y, z\right) = Ce^{-j\gamma_0(z-h)}, \quad z > h, \tag{4.1.6}$$

$$\mathcal{A}_{z2}\left(\beta_x, \beta_y, z\right) = \frac{C\left(e^{j\gamma_0 z} - \Gamma_v e^{-j\gamma_0 z}\right)}{e^{j\gamma_0 h}\left(1 - \Gamma_v e^{-j2\gamma_0 h}\right)}, \quad 0 < z < h, \tag{4.1.7}$$

$$\mathcal{A}_{z3}\left(\beta_x, \beta_y, z\right) = \frac{C\left(1 - \Gamma_v\right)e^{j\gamma z}}{e^{j\gamma_0 h}\left(1 - \Gamma_v e^{-j2\gamma_0 h}\right)}, \quad z < 0. \tag{4.1.8}$$

where $\gamma = \sqrt{k^2 - \beta^2}$ with negative imaginary part. At $z = h$, the derivative of $\mathcal{A}_z$ must be discontinuous to satisfy (4.1.3a) and the amount of discontinuity is

$$\left.\frac{\partial \mathcal{A}_{z1}}{\partial z}\right|_{z=h} - \left.\frac{\partial \mathcal{A}_{z2}}{\partial z}\right|_{z=h} = -\mu_0 \tag{4.1.9}$$

The two equations (4.1.5b), and (4.1.9) determine C and Γ_v as

$$\Gamma_v = \frac{\gamma - \kappa\gamma_0}{\gamma + \kappa\gamma_0} = 1 - \frac{\gamma_0}{\gamma_0\kappa + \gamma} 2\kappa, \tag{4.1.10a}$$

$$C = \frac{\mu_0 \left(1 - \Gamma_v e^{-j2\gamma_0 h}\right)}{2j\gamma_0}. \tag{4.1.10b}$$

The form of (4.1.10a) implies that Γ_v can be interpreted as a reflection coefficient. At distances sufficiently removed from the current element, we are interested in the solution A_{z1} which is the inverse Fourier transform of $\mathcal{A}_{z1}$, i.e.,

$$\begin{aligned} A_{z1} &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{A}_{z1} e^{-j\beta_x x} e^{-j\beta_y y} d\beta_x d\beta_y \\ &= \frac{\mu_0}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{j\gamma_0} \left(1 - \Gamma_v e^{-j2\gamma_0 h}\right) e^{-j\gamma_0 (z-h)} e^{-j\beta_x x} e^{-j\beta_y y} d\beta_x d\beta_y. \end{aligned} \tag{4.1.11}$$

This solution is more easily handled in cylindrical coordinates. For this purpose we define $\beta_x = u\cos\varphi'$, $\beta_y = u\sin\varphi'$, $x = \rho\cos\phi$, and $y = \rho\sin\phi$ and obtain

$$\begin{aligned} A_{z1} &= \frac{\mu_0}{8\pi^2} \int_{\varphi'=0}^{2\pi} \int_{u=0}^{\infty} \frac{1}{j\gamma_0} \left(1 - \Gamma_v e^{-j2\gamma_0 h}\right) e^{-j\gamma_0 (z-h)} e^{-ju\rho\cos(\phi-\varphi')} u du d\varphi' \\ &= \frac{\mu_0}{4\pi} \int_0^{\infty} \frac{1}{j\gamma_0} \left(1 - \Gamma_v e^{-j2\gamma_0 h}\right) e^{-j\gamma_0 (z-h)} J_0(u\rho)\, u du \end{aligned} \tag{4.1.12}$$

where we have made use of the fact that, [**26**],

$$\int_0^{\pi} e^{-jz\cos\phi} d\phi = \pi J_0(z). \tag{4.1.13}$$

With this change of variables we have $\beta^2 = u^2$, and therefore $\gamma_0 = \sqrt{k_0^2 - u^2}$, and $\gamma = \sqrt{k^2 - u^2}$, both with negative imaginary parts.

The result in (4.1.12) describes the solution as a spectrum of plane waves. The first term (with unity coefficient) accounts for the radiation of the current element, the other (with coefficient $\Gamma_v e^{-2\gamma_0 h}$) represents the waves reflected from the surface which appear to emanate from the image point at $z = -h$. Therefore, if we set $\Gamma_v = 0$ we should obtain the radiation of a current element in free space which means

$$\int_0^{\infty} \frac{e^{-j\sqrt{k_0^2 - u^2}(z-h)}}{j\sqrt{k_0^2 - u^2}} J_0(u\rho)\, u du = \frac{e^{-jk_0 R_1}}{R_1} \tag{4.1.14}$$

where $R_1 = \sqrt{\rho^2 + (z-h)^2}$ is the distance from the current element. This result is known as the *Sommerfeld identity*, [**27**]. Using this result (4.1.12) can be expressed as

$$\begin{aligned} A_{z1} &= \frac{\mu_0}{4\pi} \begin{pmatrix} \int_0^{\infty} \frac{e^{-j\gamma_0 (z-h)}}{j\gamma_0} J_0(u\rho)\, u du - \int_0^{\infty} \frac{e^{-j\gamma_0 (z+h)}}{j\gamma_0} J_0(u\rho)\, u du \\ -j2\kappa \int_0^{\infty} \frac{e^{-j\gamma_0 (z+h)}}{\gamma_0\kappa + \gamma} J_0(u\rho)\, u du \end{pmatrix}, \\ &= \frac{\mu_0}{4\pi} \left(\frac{e^{-jk_0 R_1}}{R_1} - \frac{e^{-jk_0 R_2}}{R_2} - j2\kappa \int_0^{\infty} \frac{e^{-j\gamma_0 (z+h)}}{\gamma_0\kappa + \gamma} J_0(u\rho)\, u du \right). \end{aligned} \tag{4.1.15}$$

where $R_2 = \sqrt{\rho^2 + (z+h)^2}$ is the distance from the image of the current element. For a perfectly conducting material, κ becomes infinity and the integral is exactly twice the term containing R_2 in the above expression giving

$$A_{z1} = \frac{\mu_0}{4\pi}\frac{e^{-jk_0R_1}}{R_1} + \frac{\mu_0}{4\pi}\frac{e^{-jk_0R_2}}{R_2}, \quad \text{as } \kappa \to \infty \tag{4.1.16}$$

which is the expected result and we have used this result in Chapter 3 with a modification that includes reflection coefficient. However, the above derivation shows that the approach we used in Chapter 3 is not exact.

The Bessel function appearing inside the integral in (4.1.15) can be written in terms of Hankel functions as $2J_0(u\rho) = H_0^{(1)}(u\rho) + H_0^{(2)}(u\rho)$. Using the fact that $H_0^{(1)}(-x) = -H_0^{(2)}(x)$ we can rewrite (4.1.15) as

$$A_{z1} = \frac{\mu_0}{4\pi}\left(\frac{e^{-jk_0R_1}}{R_1} - \frac{e^{-jk_0R_2}}{R_2} + \kappa I\right) \tag{4.1.17}$$

where

$$I = -2j\int_0^\infty \frac{e^{-j\gamma_0(z+h)}}{\gamma_0\kappa+\gamma}J_0(u\rho)\,u du = -j\int_{-\infty}^\infty \frac{e^{-j\gamma_0(z+h)}}{\gamma_0\kappa+\gamma}H_0^{(2)}(u\rho)\,u du. \tag{4.1.18}$$

Although this formulation is exact, the evaluation of the integral is necessary for a useful result.

The integrand of I has a pole when $\gamma_0\kappa+\gamma = 0$ or equivalently at $u_p = \pm k_0\sqrt{\kappa/(\kappa+1)}$. The contribution of this term to the integral I can be evaluated by the residue technique yielding a field of the form, [**25**],

$$A_z = \frac{\mu_0}{2}\frac{k}{\kappa^2-1}\sqrt{\frac{\kappa}{\kappa+1}}H_0^{(2)}\left(\frac{k\rho}{\sqrt{\kappa+1}}\right)e^{-jk_0(z+h)/\sqrt{\kappa+1}}, \tag{4.1.19}$$

which is called the *Zenneck surface wave*. At points far away from the current element and close to the surface, i.e., for $\rho \to \infty$ and $z = 0$, we find, after using the large argument asymptotic expression for the Hankel function, that

$$A_{z1} = K\frac{e^{-jk_0\rho/\sqrt{\kappa+1}}}{\sqrt{\rho}} \tag{4.1.20}$$

where K is an amplitude constant. This solution decays with $\rho^{-1/2}$ which is much slower than the usual ρ^{-1} decay rate of the free space propagation.

However, the residue term is not the dominant term in the integral. In 1919, Weyl solved the same problem with a different approach that gave a similar result but without the Zenneck surface wave term, [**28**]. Asymptotic evaluation of the integral for large ρ also shows that the Zenneck surface wave term is cancelled and what remains is a more rapidly decaying surface field which is called the *Norton surface wave*, although it is not a true surface wave but the field at the surface.

In 1930's Norton studied this problem and obtained approximate expressions that can be used in practical calculations, [**29**]. At large distances he gives the electric field intensity as

$$E_z = -j\frac{k_0 Z_0}{4\pi}\left[\cos^2\psi\left(\frac{e^{-jk_0R_1}}{R_1}+\Gamma_v\frac{e^{-jk_0R_2}}{R_2}\right)\right.$$
$$\left.+\frac{(1-\Gamma_v)}{\kappa^2}\left(\kappa^2-\kappa+\cos^2\psi\right)F\frac{e^{-jk_0R_2}}{R_2}\right], \quad (4.1.21)$$

$$E_\rho = j\frac{k_0 Z_0}{4\pi}\left[\sin\psi\cos\psi\left(\frac{e^{-jk_0R_1}}{R_1}+\Gamma_v\frac{e^{-jk_0R_2}}{R_2}\right)\right.$$
$$\left.-\frac{(1-\Gamma_v)}{\kappa}\cos\psi\sqrt{\kappa-\cos^2\psi}\left(1-\frac{(\kappa-\cos^2\psi)}{2\kappa^2}+\frac{\sin^2\psi}{2}\right)F\frac{e^{-jk_0R_2}}{R_2}\right] \quad (4.1.22)$$

where Γ_v is the Fresnel reflection coefficient for vertical polarization at a grazing angle of $\psi = \pi/2 - \theta$, given by (3.1.16a), and

$$F = 1 - j\sqrt{\pi\Omega}e^{-\Omega}\,\mathrm{erfc}\left(j\sqrt{\Omega}\right). \quad (4.1.23)$$

The complementary error function is defined as

$$\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt \quad (4.1.24)$$

and

$$\Omega = \frac{-2jk_0R_1}{(1-\Gamma_v)^2}\frac{\left(\kappa-\sin^2\theta_2\right)}{\kappa^2} \approx -jk_0R\frac{(\kappa-1)}{2\kappa^2} \quad (4.1.25)$$

The factor F is called *surface wave attenuation factor* or the *ground wave attenuation factor*. Inspection of (4.1.21) and (4.1.22) shows that the total field may be divided into two parts, a "*space wave*" which is the formula considered in the interference region, and a "*surface wave*" (which is the Norton surface wave as discussed above) that contains additional attenuation function F.

If we consider a point on the surface of the earth, the space wave term will rapidly vanish as the distance is increased, since Γ_v will approach -1 at large distances. However, the surface term will remain. This fact is experimentally verified and theoretically explained by the discussions above. On the surface of the earth, (4.1.21) and (4.1.22) reduces to

$$\mathbf{E}_{\mathrm{surface}} \approx -j\frac{k_0Z_0}{4\pi}(1-\Gamma_v)F\frac{e^{-jk_0R}}{R}\left(\frac{\kappa-1}{\kappa}\hat{\mathbf{a}}_z+\frac{\sqrt{\kappa-1}}{\kappa}\hat{\mathbf{a}}_\rho\right). \quad (4.1.26)$$

There are two important points about this result, the field will have a radial component (see Sec. 4.1.1.1), and its decay rate is determined by the surface wave attenuation factor, F. The term Ω that defines the attenuation factor is a complex quantity that is typically expressed in polar form as $\Omega = pe^{-jb}$, where $p = |\Omega|$ is known as the *numerical distance*, and b as the *phase constant*. Once p and b are determined for the given ground parameters, the attenuation factor can be calculated using (4.1.23). Computation of the attenuation function requires evaluation of the erfc (x) function for complex values. In fact, the function

$$w\,(z) = e^{-z^2}\,\mathrm{erfc}(-jz). \quad (4.1.27)$$

is called the *Fadeeva function* and its computation is easier. Numerical evaluation of the Fadeeva function is described in, [**30**]. In Fig. 4.2 the value of $|F|$ is shown as a function of the numerical distance for different values of phase constant.

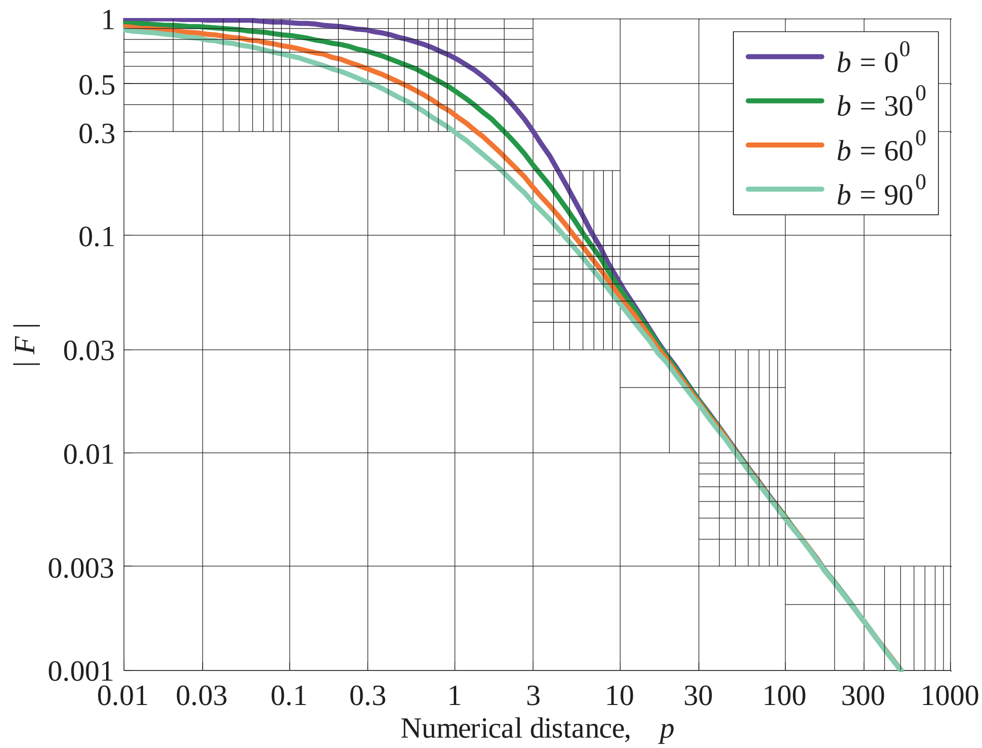

FIGURE 4.2. The flat earth surface wave attenuation factor $|F|$ as a function of the numerical distance.

Norton, [**24**], gives two approximate formulas for the calculation of $|F|$ for $b < 90\,°$, [**31**]. The first approximation is

$$f_1\left(p, b\right) = \frac{2 + 0.3p}{2 + p + 0.6p^2} - \sqrt{\frac{p}{2}}\exp\left(-5p/8\right)\sin b \tag{4.1.28}$$

and the second one is

$$f_2\left(p, b\right) = \begin{cases} e^{-0.43p+0.01p^2} - \sqrt{\dfrac{p}{2}}\exp\left(-5p/8\right)\sin b & \text{for} \quad p \leq 4.5 \\ \dfrac{1}{2p - 3.7} - \sqrt{\dfrac{p}{2}}\exp\left(-5p/8\right)\sin b & \text{for} \quad p > 4.5 \end{cases}. \tag{4.1.29}$$

However, $f_1\left(p, b\right)$ gives up to 40% error, and $f_2\left(p, b\right)$ gives up to 20% error, especially for values of $b > 60\,°$. Both formulas fail to be correct for $b = 180\,°$, [**31**].

If $\kappa' \gg 1$, which is the case for most soils, the numerical distance can be approximated as

$$p \approx \frac{k_0 R}{2\kappa} = \frac{k_0 R}{2\sqrt{(\kappa')^2 + (\kappa'')^2}}. \tag{4.1.30}$$

In the two extreme cases of $\kappa'' \gg \kappa'$ and $\kappa'' \ll \kappa'$, this result can be approximated as

$$p \approx \begin{cases} \dfrac{k_0 R}{2\kappa''} = \dfrac{\pi R}{60\lambda^2\sigma} & \text{for} \quad \kappa'' \gg \kappa' \\ \dfrac{k_0 R}{2\kappa'} = \dfrac{\pi R}{\lambda\kappa'} & \text{for} \quad \kappa'' \ll \kappa' \end{cases}. \tag{4.1.31}$$

The numerical distance depends both on the actual distance and the electrical parameters of the ground. For a perfectly conducting earth, the numerical distance reduces to zero and the attenuation factor takes the value of 1 which corresponds to free space transmission. As p increases, $|F|$ decreases which means the energy loss into the ground increases. The radio waves cannot penetrate into a well conducting surface while the situation is reversed for poorly conducting surfaces.

The frequency also plays an important role. In the VLF band ($< 300\,\text{kHz}$) ground losses are small and communication is possible up to several hundred kilometers. In the MF band (300 to 3000 kHz which includes the AM band) regular communication up to distances of 1000 km is possible. In the HF band (3 to 30 MHz), ground losses start to dominate and the surface wave distance reduces drastically. In the upper HF band the surface waves die out within a few tens of kilometers. It has been reported by CB users that while stations only 30 km apart cannot communicate, both can talk to a third station more than 1000 km apart via ionospheric skip (which will be discussed later in Chapter 7).

4.1.1.1. *The Radial Component of the Field.* The expression of the electric field at the surface given by (4.1.26) shows that there will be a horizontal (E_ρ) component of the field. This component gives rise to a Poynting vector component that points into the earth and describes the loss. The ratio of the horizontal component to the vertical component is $1/\sqrt{\kappa - 1}$. If we consider a 1 MHz radio wave at the surface of an average earth having $\sigma = 5 \times 10^{-3}$ and $\kappa' = 10$, we get

$$\frac{E_h}{E_v} = \frac{E_\rho}{E_z} = 0.105\angle 42^\circ. \tag{4.1.32}$$

Since the horizontal and vertical components are shifted in phase, the total field is elliptically polarized in a vertical plane, and the wave is no more a plane wave. The polarization ellipse lies in the plane defined by the propagation direction and the normal to the earth as shown in Fig. 4.3.

4.1.2. Horizontal Current Element. The problem of radiation from a horizontal current element over an infinite flat lossy dielectric medium can also be solved following a similar approach. The resulting field can also be decomposed into a space wave and a surface wave component. It must be noted that the space wave in this case has a small vertical component due to geometry. The surface wave will also have both horizontal and vertical components. The horizontal component is the main component, however it is shorted out by the conductive earth and the fields are much smaller as compared to a vertical dipole of same moment. The vertical component arises due to the lossy ground just like a horizontal

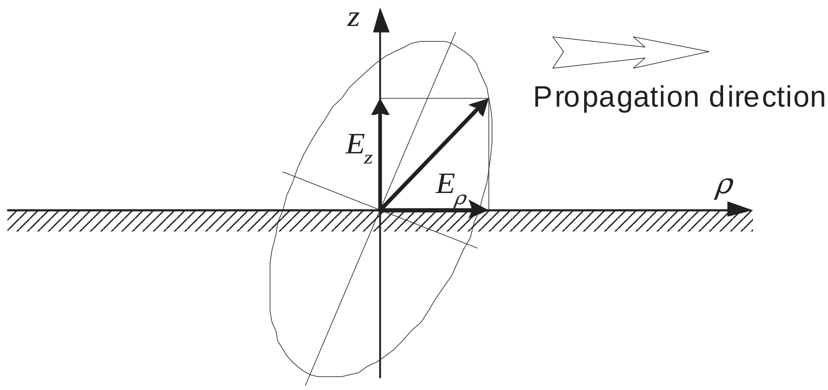

FIGURE 4.3. Polarization ellipse for a wave propagating over a surface of finite conductivity.

component arises in the case of a vertical dipole. This can be understood by the reciprocity of the fields.

As a result, the surface wave for a horizontal dipole is much smaller than a corresponding vertical dipole, and therefore this mode of propagation is not used in practice. The surface wave attenuation factor is given by the same formula (4.1.23), except that

$$\Omega = -jk_0R\frac{(\kappa - 1)}{2} \tag{4.1.33}$$

for this case.

4.2. Current Element over a Smooth Sphere

The Sommerfeld's statement of the problem overlooks the spherical nature of the earth. The real problem certainly necessitates the use of a spherical model. The problem of a dipole radiator above a lossy dielectric sphere was first solved by Watson in 1918, [**32**]. His solution is also discussed in detail in [**33**]. The electric field at the location of receiver is expressed in the form, [**34**],

$$E_{0r} = E_0 W\left(x, q\right) \tag{4.2.1}$$

where

$$E_0 = -\frac{jk_0Z_0}{4\pi}\frac{e^{-jk_0R}}{R} \tag{4.2.2}$$

is the reference field of a current element of unit moment, and $W\left(x, q\right)$ is the attenuation factor, which replaces the function $|F|$ for the flat earth case. In the present case,

$$x = \left(\frac{k_0a}{2}\right)^{1/3}\left(\frac{R}{a}\right) \tag{4.2.3}$$

is a normalized range parameter, a is the radius of earth, and

$$q = -j\left(\frac{k_0a}{2}\right)^{1/3}\Delta \tag{4.2.4}$$

where Δ is the normalized surface impedance of the ground and is given by

$$\Delta = \begin{cases} \dfrac{\sqrt{\kappa - 1}}{\kappa} & \text{for vertical polarization} \\ \sqrt{\kappa - 1} & \text{for horizontal polarization} \end{cases} . \tag{4.2.5}$$

The attenuation factor is obtained by evaluating the full-wave field integral, [**35**], which can be written in the form

$$I = \frac{1}{2} \int_C \kappa H_0^{(2)} (\kappa R) \, v (\kappa, h_t, h_r) \, d\kappa \tag{4.2.6}$$

where C is a contour that encloses all the poles of the integrand. Using the residue technique the result can be written as a series expansion given by

$$W(x, q) = 2 \left(\frac{\pi x}{j} \right)^{1/2} \sum_{n=1}^{\infty} \frac{G_n(y_t) \, G_n(y_r) \exp(-jxz_n)}{(z_n - q^2)} \tag{4.2.7}$$

where

$$G_n(y) = \frac{u(z_n - y)}{u(z_n)} = \frac{qu(z_n - y)}{u'(z_n)} \tag{4.2.8}$$

is a height-gain function, and

$$y_t = \left(\frac{2}{k_0 a} \right)^{1/3} k_0 h_t, \quad y_r = \left(\frac{2}{k_0 a} \right)^{1/3} k_0 h_r. \tag{4.2.9}$$

This result is an approximation under the assumption that the transmitter antenna height h_t and observer height h_r are both much less than the separation R between them, while the latter is much less than the earth radius, a. If a is taken as the effective earth radius, this formulation takes the atmospheric refraction into account (for the standard atmosphere).

The height-gain functions in (4.2.7) can also be approximated as $G_s(y) \approx 1 + j\Delta k_0 h$, where h is either h_t or h_r. Expanding $G_n(y)$ into a power series in y, we get

$$G_n(y) = 1 - \frac{u'(z_n)}{u(z_n)} y + O\left(y^2\right). \tag{4.2.10}$$

Using (4.2.15) in this expression gives

$$G_n(y) \approx 1 - qy \tag{4.2.11}$$

and replacing q and y from (4.2.4) and (4.2.9) gives the desired result. Notice that this result will only be valid for small values of y.

The function $u(z)$ is the Airy integral defined as

$$u(z) = \frac{1}{\sqrt{\pi}} \int_C \exp\left(zt - \frac{t^3}{3} \right) dt = \sqrt{\pi} \left[\mathrm{Bi}(z) - j \, \mathrm{Ai}(z) \right] \tag{4.2.12}$$

$$= -2\sqrt{\pi} \exp\left(\frac{5}{6} j\pi \right) \mathrm{Ai}(ze^{-j2\pi/3}) \tag{4.2.13}$$

where the contour C runs in the complex plane from $\infty e^{j2\pi/3}$ to the origin, and then along the real axis to $+\infty$. The function $u(z)$ satisfies the differential equation

$$u''(z) - zu(z) = 0. \tag{4.2.14}$$

TABLE 4.1. Typical values of surface impedance

Surface Type	$\lvert\Delta\rvert$ Vertical	$\lvert\Delta\rvert$ Horizontal
Sea water	3.3×10^{-3}	299.8
Wet ground	7.4×10^{-2}	13.5
Medium dry ground	0.204	4.8
Very dry ground	0.469	1.6

The z_n in the series expansion (4.2.7) are the poles of the integrand in (4.2.6) which are determined by the consecutive roots of the equation

$$u'(z_n) - qu(z_n) = 0, \quad n = 1, 2, \ldots \tag{4.2.15}$$

The evaluation of the attenuation factor is rather complicated. The most difficult part of the calculations is the determination of the roots from (4.2.15). Fortunately, the topic has been worked out extensively in the literature. If q is large, the poles can be approximated by the zeros of $u(t)$. Denoting these zeros by ζ_n, it can be shown that, [**35**],

$$\zeta_n = |\zeta_n| e^{-j\pi/3} \tag{4.2.16}$$

where the first few values of ζ_n are given as

$$\begin{array}{lll} |\zeta_1| = 2.33811 & |\zeta_2| = 4.08795 & |\zeta_3| = 5.52056 \\ |\zeta_4| = 6.78671 & |\zeta_5| = 7.94413 & |\zeta_6| = 9.02265 \end{array}. \tag{4.2.17}$$

When q is small, we write (4.2.15) as

$$\frac{u(z_n)}{u'(z_n)} = \frac{1}{q} \tag{4.2.18}$$

and expand the left hand side about ζ_n to obtain

$$(z_n - \zeta_n) - \frac{u''(\zeta_n)}{2u'(\zeta_n)}(z_n - \zeta_n)^2 + \cdots = \frac{1}{q}. \tag{4.2.19}$$

But from (4.2.14) we have $u''(\zeta_n) = \zeta_n u(\zeta_n) = 0$. Therefore, we may write

$$z_n = \zeta_n + \frac{1}{q} \tag{4.2.20}$$

to a first order approximation. The value of q is determined by the normalized surface impedance Δ. Typical values of Δ for $f = 1\,\text{MHz}$ are tabulated in Table (4.1).

If we assume that the permittivity and conductivity do not change with frequency, we can show that:

- The surface impedance is always larger for horizontal polarization,
- The surface impedance decreases with frequency for horizontal polarization, while it increases with frequency for vertical polarization,
- As the frequency is increased towards infinity, the surface impedance attains the limiting value of $\sqrt{\kappa' - 1}/\kappa'$ for vertical polarization and $\sqrt{\kappa' - 1}$ for horizontal polarization,
- For vertical polarization dry ground has the largest surface impedance while sea surface has the smallest,

- For horizontal polarization sea surface has the largest surface impedance while dry ground has the smallest.

The value of q increases with frequency for both vertical and horizontal polarizations. For horizontal polarization, the smallest value of q is attained for dry ground, but its value is greater than 10^3 even at frequencies as low as 10 kHz. Therefore, the correction term in (4.2.20) is negligible. However, q may take quite small values for vertical polarization, especially for propagation over sea surface in the frequency range below 100 MHz. For both horizontal and vertical polarization, correction term reduces the attenuation with range, the reduction in the case of vertical polarization being very much greater than in the case of horizontal. On this basis, the field near and beyond horizon range tends to be somewhat greater for vertical polarization than for horizontal.

The attenuation function $W(x, q)$ can also be expanded in inverse powers of $(k_0 a)$. Using the method described in, [**36**], Wait obtained the formula, [**34**],

$$W(x,q) = F(\Omega) - \left(\frac{\delta^3}{2}\right)\left[1 - j\sqrt{\pi\Omega} - (1+2\Omega)F(\Omega)\right] + \delta^6\left\{1 - j\sqrt{\pi\Omega}(1-\Omega) - 2\Omega + \frac{5}{6}\Omega^2 + \left[\frac{\Omega^2}{2} - 1\right]F(\Omega)\right\} + O\left(\delta^9\right) \quad (4.2.21)$$

where $\delta^3 = -1/(2q^3) = j/(k_0 a \Delta^3)$, $\Omega = -jk_0\Delta^2/2$ as defined in (4.1.25) and (4.1.33) before, and $F(\Omega)$ is the flat-earth attenuation function given by (4.1.23). This expansion is called the small curvature expansion and is useful in a region where the flat-earth attenuation $F(\Omega)$ is inadequate to predict the phase of the ground wave. At such points, the residue series has a very slow convergence and is difficult to handle. As the distance is increased, the residue series becomes manageable and is more accurate than (4.2.21) since the latter is a truncated power series in p which increases with distance.

It must be noted that (4.2.21) does not depend on antenna heights. Actually, this formula is similar to the flat earth surface wave attenuation factor, except that it includes the earth curvature. As $a \to \infty$, (4.2.21) reduces to flat earth formula. Thus, (4.2.21) should only be used if the antennas are sufficiently close to the earth.