EE 522 Spring 2010
Midterm Exam Solutions

1. Consider the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \\ u(x,0) &= \sin \pi x, \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0, \quad t > 0. \end{aligned}$$

Obtain a finite difference scheme by using forward difference in time (t) and central difference in space (x) coordinates. Using $h = \Delta x = 0.25$, $k = \Delta t$, and $r = k/h^2 = 0.5$, determine the value of $u(0.5, 0.125)$.

Solution With the notation $u_{pq} \triangleq u(ph, qk)$ we can write

$$\begin{aligned} \frac{\partial u}{\partial t} &\cong \frac{u_{p,q+1} - u_{pq}}{k}, \quad \text{forward difference in } t \text{ at } p,q \\ \frac{\partial^2 u}{\partial x^2} &\cong \frac{u_{p+1,q} - 2u_{pq} + u_{p-1,q}}{h^2}, \quad \text{central difference in } x \text{ at } p,q \\ \frac{u_{p,q+1} - u_{pq}}{k} &= \frac{u_{p+1,q} - 2u_{pq} + u_{p-1,q}}{h^2}, \quad \text{finite difference equation.} \end{aligned}$$

Re-arranging the finite difference equation, we get

$$u_{p,q+1} = r\left(u_{p+1,q} + u_{p-1,q}\right) + (1-2r)\, u_{pq}$$

which is an explicit formula. With $r = 0.5$, we have $k = rh^2 = 0.031\,25$, and we get

$$u_{p,q+1} = \frac{1}{2}\left(u_{p+1,q} + u_{p-1,q}\right).$$

Using this scheme we get the following result

	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
$t=0$	$\sin 0\pi = 0$	$\sin 0.25\pi = \sqrt{2}/2$	$\sin 0.5\pi = 1$	$\sin 0.75\pi = \sqrt{2}/2$	0
$t=0.031\,25$	0	$1/2$	$\sqrt{2}/2$	$1/2$	0
$t=0.062\,5$	0	$\sqrt{2}/4$	$1/2$	$\sqrt{2}/4$	0
$t=0.093\,75$	0	$1/4$	$\sqrt{2}/4$	$1/4$	0
$t=0.125$	0	$\sqrt{2}/8$	$1/4$	$\sqrt{2}/8$	0

Note that the first line is from the initial condition $u(x,0) = \sin \pi x$, and the $x = 0$ and $x = 1$ columns are from the boundary conditions $u(0,t) = u(1,t) = 0$.

The exact solution of the given BVP can be shown to be

$$u(x,t) = e^{-\pi^2 t} \sin \pi x.$$

Thus the exact value is $u(0.5, 0.125) = \left[e^{-\pi^2 t} \sin \pi x\right]_{x=0.5, t=0.125} = 0.291\,212\,933\,2$

2. If L is a positive definite, self-adjoint operator and $L\Phi = g$ has a solution Φ_0, show that the functional

$$I(\Phi) = \langle L\Phi, \Phi \rangle - 2\langle \Phi, g \rangle,$$

where Φ and g are real functions, is minimized by the solution Φ_0.

Solution Since L is positive definite, we have $\langle Lu, u \rangle \geq c^2 \|u\|$ which means $\langle Lu, u \rangle$ is real. Furthermore, since the operator is symmetric, we have $\langle Lu, u \rangle = \langle u, Lu \rangle$. Now consider $I(\Phi + \varepsilon\omega)$ where ω is a real function with $\|\omega\| = 1$ and $\varepsilon > 0$.

$$\begin{aligned}
I(\Phi + \varepsilon\omega) &= \langle L(\Phi + \varepsilon\omega), (\Phi + \varepsilon\omega) \rangle - 2\langle (\Phi + \varepsilon\omega), g \rangle \\
&= \langle (L\Phi + \varepsilon L\omega), (\Phi + \varepsilon\omega) \rangle - 2\langle (\Phi + \varepsilon\omega), g \rangle \\
&= \langle L\Phi, \Phi \rangle + \varepsilon \langle L\omega, \Phi \rangle + \varepsilon \langle L\Phi, \omega \rangle + \varepsilon^2 \langle L\omega, \omega \rangle - 2\langle \Phi, g \rangle - 2\varepsilon \langle \omega, g \rangle \\
&= \underbrace{\langle L\Phi, \Phi \rangle - 2\langle \Phi, g \rangle}_{I(\Phi)} + (\langle L\omega, \Phi \rangle + \langle L\Phi, \omega \rangle - 2\langle \omega, g \rangle)\varepsilon + \varepsilon^2 \langle L\omega, \omega \rangle
\end{aligned}$$

Thus

$$I(\Phi + \varepsilon\omega) - I(\Phi) = \delta I = (\langle L\omega, \Phi \rangle + \langle L\Phi, \omega \rangle - 2\langle \omega, g \rangle)\varepsilon + \varepsilon^2 \langle L\omega, \omega \rangle$$

To make this functional stationary we must equate the coefficient of ε to zero, i.e.:

$$\begin{aligned}
\langle L\omega, \Phi \rangle + \langle L\Phi, \omega \rangle - 2\langle \omega, g \rangle &= 0 \\
2\langle L\Phi - g, \omega \rangle &= 0, \quad \forall\omega.
\end{aligned}$$

since $\langle L\omega, \Phi \rangle = \langle \omega, L\Phi \rangle = \langle L\Phi, \omega \rangle$ (Φ and g are real functions) and $\langle \omega, g \rangle = \langle g, \omega \rangle$. Since $\langle L\Phi - g, \omega \rangle = 0$ must be satisfied for any ω, we must have $L\Phi - g = 0$ or equivalently $L\Phi = g$ which means that the solution Φ_0 of the equation $L\Phi = g$ is a stationary point for the functional $I(\Phi)$. This stationary point is a minimum since L is positive definite.

3. Consider a long hollow conductor with a uniform U-shape cross-section as shown in Fig.1.

   (a) With the mesh structure shown in the figure, write down the equations for the potentials at all internal points.

   (b) Solve the resulting system to determine the potential at point E.

   Note: You should use the symmetry of the structure.

Solution We know that the central finite difference approximation for Laplace's equation gives the potential at a point as the mean value of its four neigbors. Observing that $E = X$, $D = Y$, $C = Z$, and $W = B$ due to symmetry and using the finite difference stencil we can write

$$\begin{aligned}
E &= \frac{100 + 100 + 0 + D}{4} \\
D &= \frac{100 + E + 0 + C}{4} \\
C &= \frac{B + D + 0 + 0}{4} \\
B &= \frac{A + 100 + C + 0}{4} \\
A &= \frac{B + 100 + B + 0}{4}
\end{aligned}$$

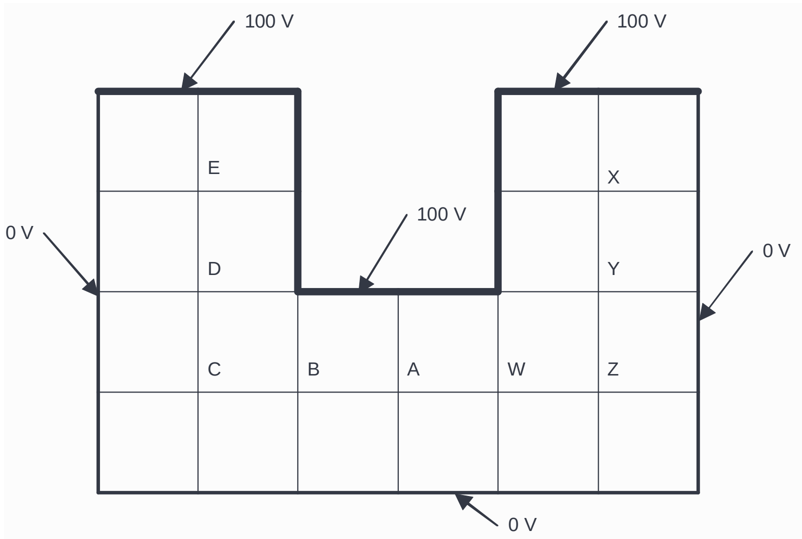

Figure 1: Figure for problem 3.

Solving this system (starting from the last) we get

$$
\begin{aligned}
A &= \frac{1}{2}B + 25 \\
B &= \left[\frac{A + 100 + C + 0}{4}\right]_{A=\frac{1}{2}B+25} \rightarrow B = \frac{250}{7} + \frac{2}{7}C \\
C &= \left[\frac{B + D + 0 + 0}{4}\right]_{B=\frac{250}{7}+\frac{2}{7}C} \rightarrow C = \frac{125}{13} + \frac{7}{26}D \\
D &= \left[\frac{100 + E + 0 + C}{4}\right]_{C=\frac{125}{13}+\frac{7}{26}D} \rightarrow D = \frac{2850}{97} + \frac{26}{97}E \\
E &= \left[\frac{100 + 100 + 0 + D}{4}\right]_{D=\frac{2850}{97}+\frac{26}{97}E} \rightarrow E = \frac{11\,125}{181}
\end{aligned}
$$

4. The integral equation

$$\int_{-w}^{w} u\left(z'\right)\left(z - z'\right) dz' = 1, \qquad -w < z < w$$

can be cast into matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

by using the Method of Moments. Using pulse basis functions and delta test functions (point matching), determine the elements of the coefficient matrix, A_{mn}, and the excitation vector, b_m. Use N equal intervals over the region $[-w, w]$ so that each interval has a length $\Delta = 2w/N$ and test the equation at the center of each interval, i.e., at $z_n = -w + \Delta\left(n - 1/2\right),$ for $n = 1, \ldots, N$.

Solution If we divide the interval $[-w, w]$ into N equal parts, each subinterval will be defined as $[-w+\Delta(n-1), -w+\Delta n]$. We define

$$p_n(z) = \begin{cases} 1 & \text{for} \quad -w+\Delta(n-1) \le z \le -w+\Delta n \\ 0 & \text{otherwise} \end{cases}$$

and write

$$u(z) = \sum_{n=1}^{N} a_n p_n(z)$$

Using this expansion in the integral equation we get

$$\begin{aligned} \int_{-w}^{w} \sum_{n=1}^{N} a_n p_n(z')(z-z')\,dz' &= 1 \\ \sum_{n=1}^{N} a_n \int_{-w}^{w} p_n(z')(z-z')\,dz' &= 1 \\ \sum_{n=1}^{N} a_n \int_{-w+\Delta(n-1)}^{-w+\Delta n} (z-z')\,dz' &= 1 \end{aligned}$$

Evaluating the integral yields

$$\sum_{n=1}^{N} a_n \left(z\Delta + w\Delta - \Delta^2 n + \frac{1}{2}\Delta^2 \right) = 1$$

We test the equation at $z_m = -w + \Delta(m-1/2)$ giving

$$\begin{aligned} \sum_{n=1}^{N} a_n \left(z_m\Delta + w\Delta - \Delta^2 n + \frac{1}{2}\Delta^2 \right) &= 1 \\ \Delta^2 \sum_{n=1}^{N} (m-n) a_n &= 1, \qquad m = 1, \ldots, N \end{aligned}$$

Thus

$$\begin{aligned} A_{mn} &= \Delta^2 (m-n) \\ b_m &= 1 \end{aligned}$$

It must be noted that the resulting matrix will be singular for $N > 2$. This implies that the solution to the matrix equation will not be unique. This is due to the fact that the solution to the integral equation is not unique, either. Two solutions are given below:

$$\begin{aligned} u(z) &= z^2 - \frac{3}{2w^3} z - \frac{1}{3} w^2 \\ u(z) &= z^3 - \frac{3}{w^2} z^2 - \frac{3}{10} \frac{2w^5+5}{w^3} z + 1 \end{aligned}$$

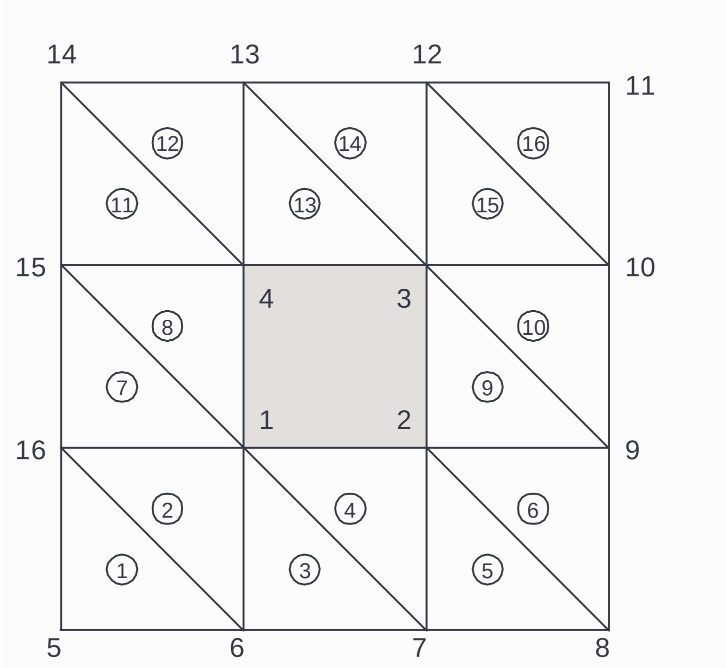

Figure 2: Figure for problem 5.

5. Consider the mesh shown in Fig.2. The shaded region is conducting and has no elements. Calculate the global FEM matrix elements $C_{3,10}$ and $C_{3,3}$ in terms of the local FEM matrices $C_{i,j}^{(e)}$, $i, j = 1, 2, 3$, $e = 1, \ldots, 16$ (Do not try to calculate local FEM matrices!). Indicate your local indexing. Do not try to impose any boundary conditions.

Solution There are two types of elements in the mesh as shown below with corresponding local indexing:

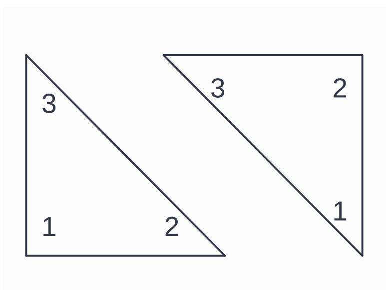

Figure 3:

With this local indexing, we can write

$$
\begin{aligned}
C_{3,10} &= C_{3,2}^{(10)} + C_{1,2}^{(15)} \\
C_{3,3} &= C_{3,3}^{(9)} + C_{3,3}^{(10)} + C_{2,2}^{(13)} + C_{1,1}^{(14)} + C_{1,1}^{(15)}
\end{aligned}
$$