Vector and Function Spaces,
Inner Product, Norm
EE 522 Spring, 2008

In this note we give the basic mathematical background on linear spaces. It is assumed that the student is familiar with three-dimensional vector analysis. The collection of vectors encountered there follows certain laws of combination which makes it an example of an algebraic system called linear spaces (or vector spaces). We first define the concept of a field. In abstract algebra, a field is an algebraic structure in which the operations of addition, subtraction, multiplication and division (except division by zero) may be performed, and the same rules hold which are familiar from the arithmetic of ordinary numbers.

Definition 1 *A field is a triplet* $(F, +, *)$ *where* F *denotes a set,* $+$ *and* $*$ *denote addition and multiplication operations that satisfy the following axioms:*

1. F *is closed under* $+$ *and* $*$*, that is for all* a*,* b *belonging to* F*, both* $a+b$ *and* $a*b$ *belong to* F *(or more formally,* $+$ *and* $*$ *are binary operations on* F*),*
2. *Both* $+$ *and* $*$ *are associative, that is for all* a*,* b*,* c *in* F*,* $a+(b+c) = (a+b)+c$ *and* $a*(b*c) = (a*b)*c$*,*
3. *Both* $+$ *and* $*$ *are commutative, that is for all* a*,* b *belonging to* F*,* $a+b = b+a$ *and* $a*b = b*a$*,*
4. *The operation* $*$ *is distributive over the operation* $+$*, that is for all* a*,* b*,* c*, belonging to* F*,* $a*(b+c) = (a*b)+(a*c)$*,*
5. *There exists an element* 0 *in* F*, such that for all* a *belonging to* F*,* $a+0 = a$*. This element is called the additive identity.*
6. *There exists an element* 1 *in* F *different from* 0*, such that for all* a *belonging to* F*,* $a*1 = a$*,*
7. *For every* a *belonging to* F*, there exists an element* $-a$ *in* F*, such that* $a+(-a) = 0$*, and* $-a$ *is called the additive inverse.*
8. *For every* $a \neq 0$ *belonging to* F*, there exists an element* a^{-1} *in* F*, such that* $a*a^{-1} = 1$*, and* a^{-1} *is called the multiplicative inverse.*

Example 2 *The set of real numbers denoted by* $\mathbb{R}$*, together with the usual addition and multiplication operations is a field.*

Example 3 *The set of complex numbers denoted by* $\mathbb{C}$*, together with the usual complex addition and complex multiplication operations is a field.*

Definition 4 *Let* F *be a field (such as the real numbers or complex numbers), whose elements will be called scalars. A vector (linear) space over the field* F *is a set* V *together with two binary operations,*

- *vector addition:* $V \times V \to V$ *denoted* $\vec{v}+\vec{w}$*, where* $\vec{v}, \vec{w} \in V$*, and*
- *scalar multiplication:* $F \times V \to V$ *denoted* $a\vec{v}$*, where* $a \in F$ *and* $\vec{v} \in V$*,*

  satisfying the axioms below. Four of the axioms require vectors under addition to form an abelian group, and two are distributive laws.

1. *Vector addition is associative, that is for all* $\vec{u}, \vec{v}, \vec{w} \in V$*, we have* $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$*.*
2. *Vector addition is commutative, that is for all* $\vec{v}, \vec{w} \in V$*, we have* $\vec{v}+\vec{w} = \vec{w}+\vec{v}$*.*
3. *Vector addition has an identity element, that is there exists an element* $\vec{0} \in V$*, called the zero vector, such that* $\vec{v}+\vec{0} = \vec{v}$ *for all* $\vec{v} \in V$*.*
4. *Vector addition has inverse elements, that is for all* $v \in V$*, there exists an element* $w \in V$*, called the additive inverse of* $\vec{v}$*, such that* $\vec{v}+\vec{w} = \vec{0}$*.*
5. *Distributivity holds for scalar multiplication over vector addition, that is for all* $a \in F$ *and* $\vec{v}, \vec{w} \in V$*, we have* $a(\vec{v}+\vec{w}) = a\vec{v}+a\vec{w}$*.*

6. *Distributivity holds for scalar multiplication over field addition, that is for all* $a, b \in F$ *and* $\vec{v} \in V$*, we have* $(a+b)\vec{v} = a\vec{v} + b\vec{v}$.

7. *Scalar multiplication is compatible with multiplication in the field of scalars, that is for all* $a, b \in F$ *and* $v \in V$*, we have* $a(b\vec{v}) = (ab)\vec{v}$.

8. *Scalar multiplication has an identity element, that is for all* $\vec{v} \in V$*, we have* $1\vec{v} = \vec{v}$*, where* 1 *denotes the multiplicative identity in* F.

Example 5 *The usual three dimensional vectors define a vector space with the usual vector addition and multiplication by a scalar.*

Example 6 *Let* $\mathbb{C}^N$ *be the set of* N*-tuples of complex numbers. An element in* $\mathbb{C}^N$ *will be denoted by* $\vec{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}$*. Let* $\vec{v}$ *and* $\vec{w}$ *be two vectors in* $\mathbb{C}^N$*. We define the addition as*

$$\vec{u} = \vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 & v_2 + w_2 & \cdots & v_N + w_N \end{bmatrix}.$$

We define multiplication with a complex number (scalar) a *as*

$$\vec{u} = a\vec{v} = \begin{bmatrix} av_1 & av_2 & \cdots & av_N \end{bmatrix}.$$

Then $\mathbb{C}^N$ *is a vector space defined over the field of complex numbers* $\mathbb{C}$.

We will drop the arrow from the vectors as a shorthand. It should be clear whether a variable is a vector or scalar from the context, and if it is not clear, we will use the arrow notation.

Example 7 *(Function space) Let* $L_2[a,b]$ *be the set of all complex valued functions defined over an interval* $[a,b]$ *that is square integrable, i.e.,*

$$L_2[a,b] = \left\{ f \mid f : [a,b] \to \mathbb{C} \text{ and } \int_a^b |f(x)|^2 \, dx < \infty \right\}$$

We can define the addition of two functions, say f *and* g *as*

$$(f+g)(x) = f(x) + g(x)$$

and multiplication by a complex number (scalar) α *as*

$$(\alpha f)(x) = \alpha f(x).$$

Then $F_{[a,b]}$ *is a vector space defined over the field of complex numbers* $\mathbb{C}$.

Definition 8 *Any subset* W *of the vector space* V *defined over the field* F *that also satisfies the vector space axioms is called a subspace of* V*, and is denoted by* $U \subset V$.

Theorem 9 *Any subset* W *of a vector space* V *defined over the field* F *that is closed under vector addition and scalar multiplication is a subspace over the same field* F*. That is if for all* u, v *in* W *and for all* a, b *in* F*,* $au + bv$ *is also in* W*, then* W *is a subspace of* V.

Definition 10 *Let* $U = \{v_i\}_{i=1}^N$ *be* N *elements in a vector space* V*. The set* U *is said to be a linearly dependent set if there exists a set of scalars* $\{\alpha_i\}_{i=1}^N$ *such that*

$$\sum_{i=1}^N \alpha_i v_i = 0. \tag{1}$$

The set U *is a linearly independent set if it is not linearly dependent, which means that (1) is satisfied if and only if* $\alpha_i = 0$ *for* $i = 1, 2, \ldots, N$.

Definition 11 *Let* $U = \{v_i\}_{i=1}^N$ *be* N *elements in a vector space* V*. The span of* U *is defined as the set of all linear combinations of its elements,*

$$Sp\{U\} = \left\{ \sum_{i=1}^{N} \alpha_i v_i \middle| \text{for any choice of } \alpha_i \in F \right\}.$$

Definition 12 *A linearly independent set that spans the space* V *is said to form a basis of* V*.*

Theorem 13 *Any basis of a vector space will have the same number of elements if it is finite. This number is called the dimension of the space.*

Example 14 *In the three-dimensional Euclidean space the three unit vectors along the coordinate axes, i.e. the set* $\{\hat{a}_x, \hat{a}_y, \hat{a}_z\}$ *is a basis. Hence the physical space is three-dimensional.*

Example 15 $\mathbb{C}^N$ *is* N *dimensional.*

Example 16 *No finite set can span the function space* $L_2[a,b]$*. It is said to be infinite dimensional. The countable set* $\left\{\sin\left(\frac{n\pi}{T}x\right), \cos\left(\frac{n\pi}{T}x\right)\right\}_{n=0}^{\infty}$ *spans the function space since any function in* $L_2[a,b]$ *can be expanded into a Fourier series.*

Theorem 17 *If* $U = \{b_i\}_{i=1}^N$ *is a basis of* V*, then any vector* v *in* V *can uniquely be written as*

$$v = \sum_{i=1}^{N} \alpha_i b_i.$$

The unique coefficients α_i *are known as the components of* v *along* b_i*.*

Definition 18 *Let* V *be a vector space defined over the field of complex numbers. A function* $\|\cdot\|$ *that maps elements of* V *into non-negative real numbers, i.e.*

$$\|\cdot\| : V \to \mathbb{R}^+$$

that satisfies the properties

1. *for all* v, w *in* V*,*
$$\|v + w\| \leqslant \|v\| + \|w\| \qquad \text{(triangle inaquality)}$$
2. *for all* v *in* V *and for all* α *in* $\mathbb{C}$
$$\|\alpha v\| = |\alpha| \|v\|$$
3. *for all* v *in* V
$$\|v\| \geqslant 0 \quad \text{and} \quad \|v\| = 0 \Leftrightarrow v = 0$$

is called a norm defined on the vector space V*.*

A norm is a function which assigns a strictly positive length (or size) to all vectors in a vector space, other than the zero vector.

Example 19 *For the vector space* $\mathbb{C}^N$ *with elements* $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}$*,*

$$\|v\| = \sum_{i=1}^{N} |v_i|^2$$

defines a norm.

Example 20 *For the function space* $L_2[a,b]$ *with elements* $f(x)$

$$\|f\| = \int_a^b |f(x)|^2 \, dx < \infty$$

defines a norm.

Definition 21 *Let V be a vector space defined over the field of complex numbers. A function $\langle \cdot, \cdot \rangle$ that maps elements of $V \times V$ into complex numbers, i.e.*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

which satisfies the properties

1. $\langle x, y \rangle = \langle y, x \rangle^*$, *for all $x, y \in V$, (conjugate symmetry),*
2. $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$, *for all $x, y, z \in V$, and for all $a, b \in \mathbb{C}$, (linearity in the first variable),*
3. $\langle x, x \rangle \geqslant 0$, *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$,
   where $(\cdot)^*$ denotes complex conjugation, is called an inner product (scalar product) defined on the vector space V.

Example 22 *Let v and w be two elements of the vector space $\mathbb{C}^N$,*

$$\langle v, w \rangle = vw^H = \sum_{i=1}^{N} v_i w_i^*$$

defines an inner product.

Example 23 *Let f and g be two elements of the function space $L_2[a, b]$,*

$$\langle f, g \rangle = \int_a^b f(x)\, g^*(x)\, dx$$

defines an inner product.

Theorem 24 *Let V be an inner-product space (i.e., a vector space with an inner product) and x be an arbitrary element of V. The function $\sqrt{\langle x, x \rangle}$ is a norm. We call this norm the natural norm of the inner-product space.*

Definition 25 *Two elements of an inner product space are said to be orthogonal if their inner product is zero,*

$$\langle x, y \rangle = 0 \Leftrightarrow x \perp y.$$

Definition 26 *Let U and W be two subspaces of an inner product space V. U and V are said to be orthogonal iff all elements in U are orthogonal to all elements in W,*

$$U \perp W \Leftrightarrow \langle x, y \rangle = 0 \qquad \forall x \in U \wedge \forall y \in W.$$

Example 27 *The $x = 0$ plane and the $z = 0$ plane are both subspaces of the three-dimensional space. These two subspaces are ortogonal.*

Definition 28 *A basis of an inner product space in which the elements are mutually orthogonal and of magnitude 1 is called an orthonormal basis.*

Definition 29 *The orthogonal complement $W^\perp$ of a subspace W of an inner product space V is the set of all vectors in V that are orthogonal to every vector in W, i.e., it is*

$$W^\perp = \{ x \in V | \langle x, y \rangle = 0 \text{ for all } y \in W \}.$$

Theorem 30 *Let W be a subspace of an inner product space V. Any element x of V can be written uniquely in the form*

$$x = x_1 + x_2$$

where $x_1 \in W$ and $x_2 \in W^\perp$. x_1 is called the orthogonal projection of x onto the subspace W. Furthermore x_1 is the vector in W that minimizes $\|x - w\|$, i.e.,

$$\min_{w \in W} \|x - w\|$$

is achieved when $w = x_1$.

Theorem 31 *Let $u_1, \ldots, u_k$ be a basis of the subspace W of an inner product space, and A be the matrix with these vectors as columns, then the projection of a vector x onto W is given by*

$$x_1 = A \left(A^H A \right)^{-1} A^H x.$$