EE 521 Fall 2007
Solution to Homework 1: Green's Function

The equation $-\varphi'' - \lambda\varphi = 0$ has no nontrivial solutions $\mathcal{L}_2^{(c)}(-\infty, \infty)$ for any λ. In fact, we have

$$\left|\exp\left(j\sqrt{\lambda}x\right)\right|^2 = e^{-2\beta x}, \quad \left|\exp\left(-j\sqrt{\lambda}x\right)\right|^2 = e^{2\beta x}.$$

Clearly for any β (including $\beta = 0$),

$$\int_{-\infty}^{\infty} \left|\exp\left(j\sqrt{\lambda}x\right)\right|^2 dx = +\infty, \quad \int_{-\infty}^{\infty} \left|\exp\left(-j\sqrt{\lambda}x\right)\right|^2 dx = +\infty.$$

Hence the equation has no eigenvalues.
If λ is not in $[0, \infty)$, we can write $\sqrt{\lambda} = \alpha + j\beta$, where we choose the branch cut for the square root as the positive real axis in λ plane. Thus for any λ we have $\beta > 0$. Let

$$g(x;\xi|\lambda) = \begin{cases} A(\xi)\exp\left(j\sqrt{\lambda}x\right) & x > \xi \\ B(\xi)\exp\left(-j\sqrt{\lambda}x\right) & x < \xi \end{cases}.$$

We have

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x;\xi|\lambda)\,dx &= \int_{-\infty}^{\xi} g(x;\xi|\lambda)\,dx + \int_{\xi}^{\infty} g(x;\xi|\lambda)\,dx \\
&= \int_{-\infty}^{\xi} B(\xi)\exp\left(-j\sqrt{\lambda}x\right)dx + \int_{\xi}^{\infty} A(\xi)\exp\left(j\sqrt{\lambda}x\right)dx \\
&= \int_{-\infty}^{\xi} B(\xi)\,e^{-jx\alpha}e^{x\beta}dx + \int_{\xi}^{\infty} A(\xi)\,e^{-x\beta}e^{jx\alpha}dx \\
&= \lim_{x\to-\infty} B(\xi)\frac{e^{\xi(-j\alpha+\beta)} - e^{-jx\alpha}e^{x\beta}}{-j\alpha+\beta} + \lim_{x\to\infty} A(\xi)\frac{e^{-\xi(-j\alpha+\beta)} - e^{-x(-j\alpha+\beta)}}{-j\alpha+\beta} \\
&= B(\xi)\frac{e^{\xi(-j\alpha+\beta)}}{-j\alpha+\beta} + A(\xi)\frac{e^{-\xi(-j\alpha+\beta)}}{-j\alpha+\beta} < \infty
\end{aligned}$$

and $g(x;\xi|\lambda)$ is in $\mathcal{L}_2^{(c)}(-\infty,\infty)$. The continuity and the jump conditions on g at $x = \xi$ are

$$\begin{aligned}
A(\xi)\,e^{j\sqrt{\lambda}\xi} - B(\xi)\,e^{-j\sqrt{\lambda}\xi} &= 0; \\
j\sqrt{\lambda}A(\xi)\,e^{j\sqrt{\lambda}\xi} + j\sqrt{\lambda}B(\xi)\,e^{-j\sqrt{\lambda}\xi} &= -1.
\end{aligned}$$

Solving these two equations for $A(\xi)$ and $B(\xi)$ we get

$$A(\xi) = \frac{1}{2}\frac{j}{\sqrt{\lambda}}e^{-j\sqrt{\lambda}\xi}; \quad B(\xi) = \frac{1}{2}\frac{j}{\sqrt{\lambda}}e^{j\sqrt{\lambda}\xi}.$$

Finally, replacing these in the expression for $g(x;\xi|\lambda)$ yields

$$g(x;\xi|\lambda) = \begin{cases} \frac{1}{2}\frac{j}{\sqrt{\lambda}}e^{j\sqrt{\lambda}(x-\xi)} & x > \xi \\ \frac{1}{2}\frac{j}{\sqrt{\lambda}}e^{j\sqrt{\lambda}(\xi-x)} & x < \xi \end{cases}$$

which can be written as

$$g(x;\xi|\lambda) = \frac{1}{2}\frac{j}{\sqrt{\lambda}} e^{j\sqrt{\lambda}(x_> - x_<)} = \frac{1}{2}\frac{j}{\sqrt{\lambda}} e^{j\sqrt{\lambda}x_>} e^{-j\sqrt{\lambda}x_<} = \frac{j}{2\sqrt{\lambda}} e^{j\sqrt{\lambda}|x-\xi|}.$$

The function g has a branch along the positive real λ axis. Directly above and on the positive real axis, we have

$$g_+ = \frac{j}{2|\lambda|^{1/2}} e^{j|\lambda|^{1/2}|x-\xi|},$$

and directly below the positive real axis

$$g_- = \frac{j}{-2|\lambda|^{1/2}} e^{-j|\lambda|^{1/2}|x-\xi|}.$$

Thus the jump $[g]$ as we cross the positive real λ axis is

$$\begin{aligned} [g] &= g_+ - g_- = \frac{j}{2|\lambda|^{1/2}} e^{j|\lambda|^{1/2}|x-\xi|} - \frac{j}{-2|\lambda|^{1/2}} e^{-j|\lambda|^{1/2}|x-\xi|} \\ &= j\frac{\cos\sqrt{\lambda}(x-\xi)}{\sqrt{\lambda}} = j\frac{\cos\sqrt{\lambda}\xi\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}\xi\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \end{aligned}$$

Now consider the contour C that consists of a large circle and C_+ and C_- as shown in Fig. (1) below. Since g is analytic except on the positive real λ axis, we have

$$\int_C g\,d\lambda = 0.$$

On the other hand,

$$\int_C g\,d\lambda = \oint g\,d\lambda + \int_{C_+} g\,d\lambda + \int_{C_-} g\,d\lambda.$$

Since on C_-, λ ranges from $+\infty$ to 0, we have

$$\int_C g\,d\lambda = \oint g\,d\lambda + \int_0^\infty [g]\,d\lambda$$

where $[g]$ is as calculated above.
Using

$$\oint g(x;\xi|\lambda)\,d\lambda = -2\pi j\delta(x-\xi)$$

we obtain

$$\begin{aligned} \delta(x-\xi) &= \frac{1}{2\pi j}\int_0^\infty [g]\,d\lambda \\ &= \frac{1}{2\pi}\int_0^\infty \frac{\cos\sqrt{\lambda}\xi\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}\xi\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\,d\lambda \end{aligned}$$

Finally, by making a change of variables with $\lambda = v^2$

$$\delta(x-\xi) = \frac{1}{\pi}\int_0^\infty [\cos v\xi\cos vx + \sin v\xi\sin vx]\,dv.$$

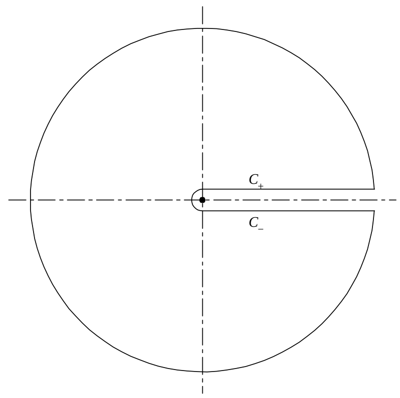

Figure 1: The integration contour.

Multiplying this result by $f(x)$ on both sides and integrating over $(-\infty, \infty)$ we get

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-\xi)\,dx = \int_{-\infty}^{\infty} f(x)\,\frac{1}{\pi}\int_{0}^{\infty}\left[\cos v\xi\cos vx + \sin v\xi\sin vx\right]\,dv dx$$

or equivalently

$$f(\xi) = \frac{1}{\pi}\int_{0}^{\infty}\cos v\xi\left[\int_{-\infty}^{\infty} f(x)\cos vx\,dx\right]\,dv + \frac{1}{\pi}\int_{0}^{\infty}\sin v\xi\left[\int_{-\infty}^{\infty} f(x)\sin vx\,dx\right]\,dv.$$

By using the trigonometric identity

$$\cos v\xi\cos vx + \sin v\xi\sin vx = \cos v(x-\xi),$$

the expansion

$$\delta(x-\xi) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos v\xi\cos vx + \sin v\xi\sin vx\right]\,dv$$

can also be written as

$$\delta(x-\xi) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos v(x-\xi)\right]\,dv.$$

Since the cosine function is even and the sine function is odd, we can also write

$$\begin{aligned}\delta(x-\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\cos v(x-\xi)\right]\,dv + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[j\sin v(x-\xi)\right]\,dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jv(x-\xi)}\,dv.\end{aligned}$$

Again, multiplying by $f(x)$ on both sides and integrating over $(-\infty, \infty)$ we get

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-\xi)\,dx = \int_{-\infty}^{\infty} f(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{jv(x-\xi)}\right]\,dv\right)\,dx$$

or equivalently,

$$f(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jv\xi}\left(\int_{-\infty}^{\infty} f(x)\,e^{jvx}\,dx\right)\,dv.$$

Now consider the system

$$-u'' - \lambda u = f(x).$$

The function $\int_{-\infty}^{\infty} g(x;\xi|\lambda) f(\xi)\, d\xi$ satisfies the differential equation since

$$\begin{aligned}
\left(-\frac{d^2}{dx^2} - \lambda\right) \int_{-\infty}^{\infty} g(x;\xi|\lambda) f(\xi)\, d\xi &= \int_{-\infty}^{\infty} \left(-\frac{d^2}{dx^2} - \lambda\right) g(x;\xi|\lambda) f(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} \delta(x-\xi) f(\xi)\, d\xi = f(x).
\end{aligned}$$

We also need to show that this solution is indeed in $\mathcal{L}_2^{(c)}(-\infty,\infty)$. To this extend, first consider

$$\begin{aligned}
|u(x)|^2 &= \left|\int_{-\infty}^{\infty} g(x;\xi|\lambda) f(\xi)\, d\xi\right|^2 = \left|\int_{-\infty}^{\infty} g(x;\xi|\lambda) f(\xi)\, d\xi\right| \times \left|\int_{-\infty}^{\infty} g(x;\eta|\lambda) f(\eta)\, d\eta\right| \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x;\xi|\lambda)|\, |g(x;\eta|\lambda)|\, |f(\xi)|\, |f(\eta)|\, d\xi\, d\eta.
\end{aligned}$$

Since $|f(\xi)|\, |f(\eta)| \le \frac{1}{2}\left(|f(\xi)|^2 + |f(\eta)|^2\right)$,

$$|u(x)|^2 \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x;\xi|\lambda)|\, |g(x;\eta|\lambda)|\, |f(\xi)|^2\, d\xi\, d\eta.$$

Now, we also have

$$\int_{-\infty}^{\infty} |g(x;\xi|\lambda)|\, d\xi \le l = \frac{1}{|\lambda|^{1/2}\beta}, \qquad \int_{-\infty}^{\infty} |g(x;\xi|\lambda)|\, dx \le l = \frac{1}{|\lambda|^{1/2}\beta}.$$

which is found by direct integration. Then

$$\|u\|^2 = \int_{-\infty}^{\infty} |u(x)|^2\, dx \le l^2 \int_{-\infty}^{\infty} |f(\xi)|^2\, d\xi = l^2 \|f\|^2$$

and hence the solution is in $\mathcal{L}_2^{(c)}(-\infty,\infty)$. Finally, any solution of $-u''-\lambda u = f(x)$ can differ from each other by the solution of the homogeneous equation. But, since the homogeneous equation has no solution in $\mathcal{L}_2^{(c)}(-\infty,\infty)$ as shown above, the function

$$u(x) = \int_{-\infty}^{\infty} g(x;\xi|\lambda) f(\xi)\, d\xi$$

is the unique solution.
Now consider the inhomogeneous equation

$$-u'' - \lambda u = f(x).$$

Multiply both sides by e^{jvx} and integrate from $-\infty$ to ∞.

$$\int_{-\infty}^{\infty} (-u'' - \lambda u)\, e^{jvx} dx = \int_{-\infty}^{\infty} f(x)\, e^{jvx} dx.$$

Since u and u' must vanish at $\pm\infty$, we obtain, after integrating the first term by parts twice,

$$v^2 U(v) - \lambda U(v) = F(v)$$

where

$$U(v) = \int_{-\infty}^{\infty} u(x) e^{jvx} dx; \quad F(v) = \int_{-\infty}^{\infty} f(x) e^{jvx} dx.$$

Thus,

$$U(v) = \frac{F(v)}{v^2 - \lambda}$$

and

$$\begin{aligned} u(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U(v) e^{-jvx} dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(v)}{v^2 - \lambda} e^{-jvx} dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{v^2 - \lambda} e^{-jvx} \left(\int_{-\infty}^{\infty} f(x) e^{jvx} dx \right) dv \end{aligned}$$

In the particular case where $f(x) = \delta(x - \xi)$, we get

$$\begin{aligned} g(x; \xi | \lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{v^2 - \lambda} e^{-jvx} \left(\int_{-\infty}^{\infty} \delta(x - \xi) e^{jvx} dx \right) dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-jv(x-\xi)}}{v^2 - \lambda} dv. \end{aligned}$$