EE 521 Fall 2007
Midterm Exam 1: Take Home (Open books)

The due date is Tuesday December 4.

1. Consider the differential equation

$$-\frac{d}{dx}\left(x\frac{du}{dx}\right)-\lambda\frac{u}{x}=0$$

over the interval $0<x<\infty$.

(a) Determine the two independent solutions of this equation. (Hint: Assume a solution of the form $u=x^{\alpha}$).

(b) Assuming that λ is not in $[0,\infty)$, construct the Green's function $g\left(x;\xi|\lambda\right)$ which satisfies

$$-\frac{d}{dx}\left(x\frac{dg}{dx}\right)-\lambda\frac{g}{x}=\delta\left(x-\xi\right);\quad 0<x,\xi<\infty;\quad \int_0^\infty\frac{|g|^2}{x}dx<\infty.$$

(c) Using

$$\frac{1}{2\pi j}\oint g\left(x;\xi|\lambda\right)d\lambda=-x\delta\left(x-\xi\right)$$

obtain the formulas

$$\begin{aligned} f\left(x\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}x^{j\nu}F_M\left(\nu\right)\,dv, \\ F_M\left(\nu\right) &= \int_0^\infty\frac{f\left(x\right)}{x}x^{-j\nu}\,dx. \end{aligned}$$

The function $F_M\left(\nu\right)$ is known as the Mellin transform of f and the first line of the above equation is the inversion formula.

(d) Using these results, obtain the bilinear expansion for the Green's function.

2. Consider the Laplace's equation

$$\nabla^2\Phi=0.$$

(a) Obtain the eigenfunction expansion for the free-space Green's function for the Laplace's equation using the separation of variables technique in spherical coordinates. The Green's function is required to be finite everywhere (except possibly at the source point).

(b) Translating the coordinate origin to the source point, show that

$$G_0\left(\mathbf{r};\mathbf{r}'\right)=\frac{1}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}.$$

(c) Obtain the eigenfunction expansion for the Laplace's equation for the "exterior" problem with a spherical boundary at $r=a$. The Green's function must be finite everywhere and must satisfy the Dirichlet boundary condition on the sphere, i.e., $G\left(\mathbf{r};\mathbf{r}'\right)=0$ for $|\mathbf{r}|=r=a$.

(d) Show that the result in (c) is equivalent to

$$G\left(\mathbf{r};\mathbf{r}'\right)=G_0\left(\mathbf{r};\mathbf{r}'\right)-\frac{a}{r'}G_0\left(\mathbf{r};\frac{a^2}{r'^2}\mathbf{r}'\right).$$