EE 521 Fall 2007
Homework 1: Green's Function

The due date for this assignment is Tuesday October 30.

1. Over the interval $-\infty < x < \infty$, consider the three systems

$$
\begin{aligned}
-\varphi'' - \lambda\varphi &= 0; && \text{(1a)}\\
-u'' - \lambda u &= f(x); && \text{(1b)}\\
-\frac{d^2 g}{dx^2} - \lambda g &= \delta(x-\xi). && \text{(1c)}
\end{aligned}
$$

We are looking for solutions in $\mathcal{L}_2^{(c)}(-\infty,\infty)$ and assume that f is in $\mathcal{L}_2^{(c)}(-\infty,\infty)$. Show that (1a) has no eigenvalues. Show that if λ is not in $[0,\infty)$,

$$g(x;\xi|\lambda) = \frac{j}{2\sqrt{\lambda}}\exp\left(j\sqrt{\lambda}x_>\right)\exp\left(-j\sqrt{\lambda}x_<\right) = \frac{j}{2\sqrt{\lambda}}\exp\left(j\sqrt{\lambda}\,|x-\xi|\right).$$

Using

$$\frac{1}{2\pi j}\oint g(x;\xi|\lambda)\,d\lambda = -\sum_n \varphi_n(x)\,\overline{\varphi}_n(\xi) = -\frac{\delta(x-\xi)}{w(x)},$$

obtain the expansion

$$\delta(x-\xi) = \frac{1}{\pi}\int_0^\infty [\cos vx\cos v\xi + \sin vx\sin v\xi]\,dv, \tag{2}$$

from which one deduces the trigonometric form of the Fourier integral theorem

$$f(\xi) = \frac{1}{\pi}\int_0^\infty \cos v\xi\left[\int_{-\infty}^\infty f(x)\cos vx dx\right]dv + \frac{1}{\pi}\int_0^\infty \sin v\xi\left[\int_{-\infty}^\infty f(x)\sin vx dx\right]dv.$$

Show that (2) can be rewritten as

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{jv(x-\xi)}dv,$$

from which the exponential form of the Fourier integral theorem devolves:

$$f(\xi) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-jv\xi}\left[\int_{-\infty}^\infty f(x)\,e^{jvx}dx\right]dv.$$

Show that if λ is not in $[0,\infty)$, system (1b) has a unique $\mathcal{L}_2^{(c)}(-\infty,\infty)$ solution

$$u(x) = \int_{-\infty}^\infty g(x;\xi|\lambda)\,f(\xi)\,d\xi.$$

Express $u(x)$ as a Fourier integral (exponential form) and obtain the bilinear expansion for g.