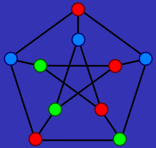
Graph Theory, Part 1



Lecture Notes in Math 212 Discrete Mathematics

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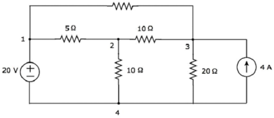
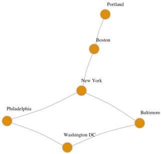
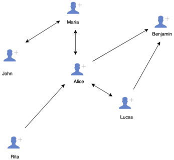
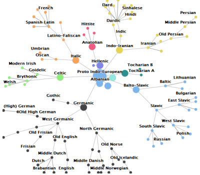
The idea of graph



People often use schemes where some objects are marked by nodes connected with links. Examples: transport networks, electrical circuits, etc.

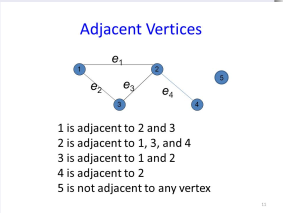






Mathematically significant information in these examples is the **graph**

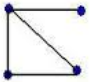
formed by a set of nodes, called vertices (usually marked as points on a plane) and a set of links called edges (drawn as lines connecting some vertices). Every edge connects two vertices called the endpoints of this edge. The endpoints are said to be incident to the edge. Vertices connected by an edge are called adjacent, or neighbors.



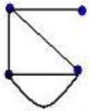
Variants of graphs

Simple graphs and other kinds

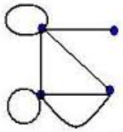
- If some pairs of points are connected by several edges, such kind of a graph is called multigraph.
- If in addition to multiple edge a graph is allowed to contain loop-edges connecting a vertex with itself, it is called pseudograph.
- If a graph has neither multiple edges nor loops, it is called simple graphs. Later on we usually suppose that the graphs are simple.
- If in a graph every edge is assigned a direction, it is called directed graph or digraph.



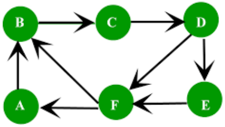
simple graph



multigraph



pseudograph

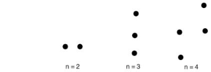


Directed Graph

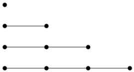
Important examples

We suppose by definition that a graph contains at least one vertex, that is, its vertex set V is non-empty. But the set of edges E can be empty.

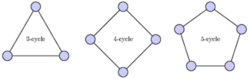
- If $E = \varnothing$, the graph is called empty or nullgraph.



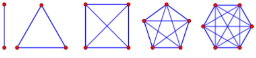
- An n-path graph P_n has n vertices $v_1, \dots, v_n$ and edges connecting v_i with v_{i+1}, $i = 1, \dots, n-1$.



- An n-cycle graph C_n, the vertices are cyclically connected like in polygon: v_i to v_{i+1} and v_n to v_1.

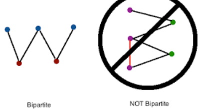


- A graph is said to be complete if every pair of its vertices are connected by an edge (in other words, all vertices are adjacent to each other). A complete graph with n vertices will be denoted K_n.



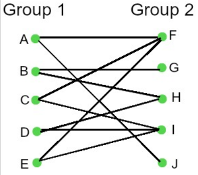
Complete Graphs

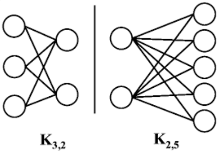
Question: Find the number of edges in K_n. **Answer:** $\binom{n}{2} = \frac{n(n-1)}{2}$, which is the number of pairs of points.



A graph is called bipartite graph if its vertices

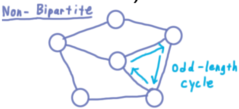
can be split in two groups (or can be colored in two colors), so that every edge have endpoints in different sets (of different colors).
In a complete bipartite graph every pair of vertices from different groups must be connected by an edge. Such graph is denoted $K_{m,n}$, where m and n are the numbers of vertices in the two groups.



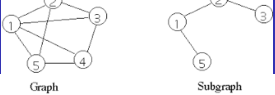


- How many edges are in K_{mn} ? **Answer:** mn.
- Are path graphs P_n and cycle graphs C_n bipartite ? **Answer.** P_n is bipartite for any n: one group of vertices are $v_1, v_3, \ldots$ (odd indices) and another groups is $v_2, v_4, \ldots$ (even indices). C_n is bipartite for even n and not for odd n (since the colors of vertices alternate).

A bipartite graph cannot contain triangles and more generally, odd-length cycles.



Subgraphs

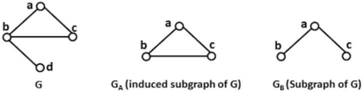


Graphs are denoted as pairs $G = (V, E)$, where V and E are the sets of vertices and edges.

A subgraph of graph $G = (V, E)$ is a graph $G_1 = (V_1, E_1)$ such that

$V_1 \subset V$ and $E_1 \subset E$. Warning: it is important to check that for every edge in E_1 its endpoints belong to V_1, otherwise G_1 is not a graph.

- G_1 is called spanning subgraph if $V_1 = V$.
- G_1 is called subgraph induced by subset of vertices V_1 if it includes all the edges whose endpoints belong to V_1.



Exercises: prove that

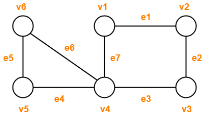
- if a subgraph G_1 of graph G is spanning and induced, then $G_1 = G$;
- a subgraph of a bipartite graph is also bipartite.

Paths and cycles in a graph

A path or a cycle in a graph can be represented by subgraphs which are path graphs and cycle graphs respectively. Formal definitions:

A walk in a graph is a sequence of consecutive vertices

linked by edges: $v_1, e_1, v_2, e_2 \dots, e_{n-1}, v_n$, where edge e_i connects v_i with v_{i+1}. A walk is called closed if $v_n = v_1$.



For simple graphs the edges are determined by their endpoints and a walk can be denoted just $v_1, \dots, v_n$.

- A trail is a kind of a walk without repetitions of edges. A closed trail is called a cirquit.
- A path is a kind of a trail (walk) without repetitions of vertices. A closed path is called a cycle.

	Vertices	Edges	
Walks	Repetition allowed	Repetition allowed	
Trails	Repetition allowed	No repetition of edges	
Paths	No repetition of vertices except possibly starting and terminal vertices	No repetition of edges	
Circuits	Repetition allowed	No repetition of edges	A nontrivial closed trail
Cycles	No repetition of vertices except starting and terminal vertices	No repetition of edges	A nontrivial closed trail without repetition of vertices except starting and terminal vertices

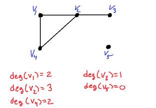
Degree (valency) of a vertex

Degree or valency of a vertex v in a graph

is the number of edges incident to this vertex, notation deg(v). In a simple graph it is the same as the number of vertices adjacent to v.

Exercise: give an example of multigraph for which it is not the same.

- vertices of degree 0 are called isolated
- vertices of degree 1 are called pendant.



Hand-shaking theorem

The sum of degrees of all vertices in a graph $G = (V, E)$ (possibly multigraph) equals to the double number of its edges: $\sum_{v \in V} deg(v) = 2|E|$.

Proof. If an edge has endpoints v and w, then it contributes 2 to the sum $\sum_{v \in V} deg(v)$: 1 in summand $deg(v)$ and 1 in $deg(w)$.

Corollary: in a graph there are even number of vertices of odd degree.

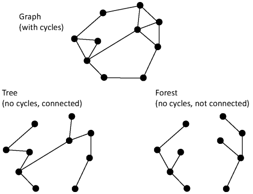
Proof: because the sum of all the degrees is even.

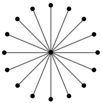
Trees

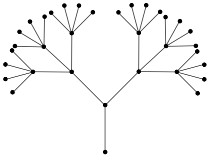
A graph is said to be connected if any pair of its vertices can be connected by a path (equivalently, by a trail or a walk).

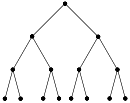
A tree is a connected graph that does not contain cycles

A graph without cycles, but not connected is called forest. It contains several connected components, which are trees.





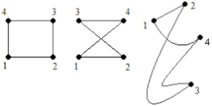




- In a tree $v = e + 1$, where v, e are the numbers of vertices and edges.
- In a tree every pair of edges can be linked by a unique path.
- Pendant vertices (of degree 1) in a tree are called leaves.
- A tree has at least two leaves.

Isomorphism of graphs

Informally speaking, isomorphic graphs differ just by the way of their presentation in the plane: **vertices and edges may change their position, but cannot appear or disappear.**

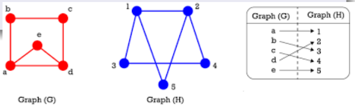


First, assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are simple graphs.

An isomorphism between graphs G_1 and G_2

is a bijective correspondence between their vertices, $f : V_1 \to V_2$, such that a pair of vertices $v, w \in V_1$ are adjacent in G_1 if and only if their images $f(v)$, $f(w)$ are adjacent in G_2.

Bijection f is defined by a list of correspondence between sets V_1 and V_2

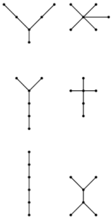


In the definition of isomorphism for multigraphs G_1 and G_2

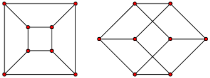
we require in addition to f a bijection between edges $\tilde{f} : E_1 \to E_2$, such that that a vertex $v \in V_1$ and edge $e \in E_1$ are incident if and only if $f(v)$ and $\tilde{f}(e)$ are incident.

Examples of isomorphism

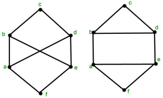
Example: six non-isomorphic trees with six vertices. In all examples except two, non-isomorphism is proved by counting the number of vertices of different degrees. How to prove that two exceptional examples are not isomorphic ? Note that in one of then the vertices of degree 2 are adjacent, and in the other are not.



Exercise: show isomorphism of two graphs by labeling their corresponding vertices with $1, \dots, 8$



How to prove non-isomorphism of these two graphs ?
Note that one of them is bipartite and the other is not !



Exercise: prove or disprove isomorphisms of the following graphs.