TUTORIAL PROBLEMS ON KINEMATICS
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PROBLEM 1

Figure 1 shows a helicopter moving with respect to an earth-fixed frame \( \mathcal{F}_e(O) \). Let the frames attached to the helicopter and its rotor be \( \mathcal{F}_h(C) \) and \( \mathcal{F}_r(Q) \) respectively. The orientational relationships involving these frames are described as follows:

\[
\mathcal{F}_e \xrightarrow{\hat{u}_3^{(e)}} \mathcal{F}_m \xrightarrow{\hat{u}_2^{(m)}} \mathcal{F}_n \xrightarrow{\hat{u}_1^{(n)}} \mathcal{F}_h \xrightarrow{\hat{u}_3^{(h)}} \mathcal{F}_r
\]

Let the dyads \( \bar{R}_{r/h} \) and \( \bar{R}_{r/e} \) represent the rotation operators that orient the rotor with respect to the helicopter and the earth.

a) Obtain the transformation matrices \( \hat{C}^{(e,h)} \), \( \hat{C}^{(h,r)} \), and \( \hat{C}^{(e,r)} \).

b) Obtain the matrix representations of \( \bar{R}_{r/h} \) and \( \bar{R}_{r/e} \) in \( \mathcal{F}_h(C) \).

c) Obtain the matrix representations of \( \bar{R}_{r/h} \) and \( \bar{R}_{r/e} \) in \( \mathcal{F}_e(O) \).

Express all the required matrices using exponential rotation matrices and their products.

d) The unit vector \( \hat{w} \) (along one of the rotor blades) is obtained by rotating \( \hat{w}_0/\hat{u}_1^{(e)} \) (the first basis vector of \( \mathcal{F}_e \)) with \( \bar{R}_{er} = \bar{R}_{r/e} \). Formulate this operation separately as observed in \( \mathcal{F}_h \) and \( \mathcal{F}_e \). Thus, express \( \hat{w}^{(h)} = \{\hat{w}\}^{(h)} \) and \( \hat{w}^{(e)} = \{\hat{w}\}^{(e)} \) using the results of the previous parts. Check your expressions by verifying the fact that \( \hat{w}^{(e)} = \hat{C}^{(e,h)} \hat{w}^{(h)} \).

SOLUTION

a) \( \hat{C}^{(e,h)} = \hat{C}^{(e,m)} \hat{C}^{(m,n)} \hat{C}^{(n,h)} = e^{\hat{u}_3 \gamma} e^{\hat{u}_2 \theta} e^{\hat{u}_1 \phi} \).

\( \hat{C}^{(h,r)} = e^{\hat{u}_3 \gamma} \).

\( \hat{C}^{(e,r)} = \hat{C}^{(e,h)} \hat{C}^{(h,r)} = e^{\hat{u}_3 \gamma} e^{\hat{u}_2 \theta} e^{\hat{u}_1 \phi} e^{\hat{u}_3 \gamma} \).

b) \( \{\bar{R}_{r/h}\}^{(h)} = \bar{K}_{hr} = \hat{C}^{(h,r)} = e^{\hat{u}_3 \gamma} \).

M. K. Ö zgören
\[ \{ \dot{R}_{r/e} \}^{(h)} = \hat{R}_{er}^{(h)} = \hat{C}^{(h,e)} \hat{R}_{er}^{(e)} \hat{C}^{(e,h)} = \hat{C}^{(h,e)} \hat{C}^{(e,r)} \hat{C}^{(e,h)} = \hat{C}^{(h,r)} \hat{C}^{(e,h)} \]
\[ = e^{i\bar{\alpha}_3 \gamma} e^{i\bar{\alpha}_3 \psi} e^{i\bar{\alpha}_2 \theta} \bar{u}_3 e^{i\bar{\alpha}_2 \theta} e^{i\bar{\alpha}_1 \phi}. \]

\( c) \ \{ \dot{R}_{r/h} \}^{(e)} = \hat{R}_{hr}^{(e)} = \hat{C}^{(e,h)} \hat{R}_{hr}^{(h,e)} = \hat{C}^{(e,h)} \hat{C}^{(h,r)} \hat{C}^{(h,e)} \]
\[ = e^{i\bar{\alpha}_3 \psi} e^{i\bar{\alpha}_2 \theta} e^{i\bar{\alpha}_1 \phi} e^{i\bar{\alpha}_3 \gamma} e^{-i\bar{\alpha}_1 \phi} e^{i\bar{\alpha}_2 \theta} e^{-i\bar{\alpha}_3 \Psi}. \]

\[ \{ \dot{R}_{r/e} \}^{(e)} = \hat{R}_{er}^{(e)} = \hat{C}^{(e,r)} \hat{R}_{er}^{(e)} \hat{C}^{(e,h)} = \hat{C}^{(e,r)} \hat{C}^{(e,h)} \]
\[ = e^{i\bar{\alpha}_3 \psi} e^{i\bar{\alpha}_2 \theta} e^{i\bar{\alpha}_1 \phi} e^{i\bar{\alpha}_3 \gamma}. \]

\( d) \ \ddot{w} = \ddot{R}_{er} \cdot \dot{u}_1^{(e)}. \]
\[ \ddot{w}^{(h)} = \ddot{R}_{er}^{(h)} \cdot \dot{u}_1^{(e/h)} = \ddot{R}_{er}^{(h)} \cdot \dot{u}_1^{(e/e)} = \ddot{R}_{er}^{(h)} \cdot \dot{u}_1^{(h/e)} = [\ddot{C}^{(h,r)} \ddot{C}^{(e,h)} \ddot{u}_1^{(e)}] = [\ddot{C}^{(h,r)}] \ddot{C}^{(e,h)} \ddot{u}_1^{(e)} \]
\[ = \dot{C}^{(h,r)} \ddot{u}_1^{(e)} = e^{i\bar{\alpha}_3 \gamma} \ddot{u}_1^{(e)}; \]
\[ \ddot{w}^{(e)} = \ddot{R}_{er}^{(e)} \cdot \dot{u}_1^{(e/e)} = \ddot{R}_{er}^{(e)} \cdot \dot{u}_1^{(e)} = \ddot{C}^{(e,r)} \ddot{u}_1^{(e)} \]
\[ = e^{i\bar{\alpha}_3 \psi} e^{i\bar{\alpha}_2 \theta} e^{i\bar{\alpha}_1 \phi} e^{i\bar{\alpha}_3 \gamma} \ddot{u}_1^{(e)}. \]

Verification:
\[ \ddot{w}^{(e)} = \ddot{C}^{(e,h)} \ddot{w}^{(h)} = [e^{i\bar{\alpha}_3 \psi} e^{i\bar{\alpha}_2 \theta} e^{i\bar{\alpha}_1 \phi} e^{i\bar{\alpha}_3 \gamma} \ddot{u}_1^{(e)}] = e^{i\bar{\alpha}_3 \psi} e^{i\bar{\alpha}_2 \theta} e^{i\bar{\alpha}_1 \phi} e^{i\bar{\alpha}_3 \gamma} \ddot{u}_1^{(e)}. \] (Checks!)

**PROBLEM 2**

Consider the same helicopter described in Problem 1.

a) Express \( \ddot{\omega}_{r/lh}^{(h)} \) = \( \{ \ddot{\omega}_{r/lh} \}^{(h)} \).

b) Express \( \ddot{\omega}_{r/le}^{(h)} \) = \( \{ \ddot{\omega}_{r/le} \}^{(h)} \).

c) Express \( \ddot{\alpha}_{r/le}^{(h)} \) = \( \{ \ddot{\alpha}_{r/le} \}^{(h)} \).

**SOLUTION**

a) \( \ddot{\omega}_{r/lh} = \dot{\gamma} \ddot{u}_3^{(h)} \)
\[ \Rightarrow \ddot{\omega}_{r/lh}^{(h)} = \dot{\gamma} \ddot{u}_3. \]

b) \( \ddot{\omega}_{r/le} = \ddot{\omega}_{r/lh} + \ddot{\omega}_{h/lh} + \ddot{\omega}_{n/m} + \ddot{\omega}_{m/e} = \dot{\gamma} \ddot{u}_3^{(h)} + \ddot{\phi} \ddot{u}_1^{(h)} + \ddot{\theta} \ddot{u}_2^{(h)} + \ddot{\psi} \ddot{u}_3^{(h)}; \)
\[ \ddot{\omega}_{r/le}^{(h)} = \ddot{\omega}_{r/lh}^{(h)} + \ddot{\phi} \ddot{u}_1^{(h)} + \ddot{\theta} \ddot{u}_2^{(h)} + \ddot{\psi} \ddot{u}_3^{(h)}, \]
\[ \ddot{\omega}_{r/le}^{(h)} = \ddot{\omega}_{r/e}^{(h)} + \dot{\phi} \ddot{u}_1^{(h)} + \dot{\theta} \ddot{C}^{(h,n)} \ddot{u}_2^{(h)} + \dot{\psi} \ddot{C}^{(h,m)} \ddot{u}_3^{(h)}, \]
\[ \ddot{\omega}_{r/le}^{(h)} = \ddot{\omega}_{r/lh}^{(h)} + \dot{\phi} \ddot{u}_1^{(h)} + \dot{\theta} \ddot{C}^{(h,n)} \ddot{u}_2^{(h)} + \dot{\psi} \ddot{C}^{(h,m)} \ddot{u}_3^{(h)} + \ddot{\psi} \ddot{u}_3^{(h)} + \ddot{\phi} \ddot{u}_1^{(h)} \]
\[ = \ddot{u}_1 (\dot{\phi} - \dot{\psi} s \theta) + \ddot{u}_2 (\dot{\theta} c \phi + \dot{\psi} s c \phi \theta) + \ddot{u}_3 (\dot{\gamma} - \dot{\theta} s \phi + \dot{\psi} c c \phi \theta). \]
c) \( \vec{\alpha}_{r/e} = D_r\vec{\omega}_{r/e} = D_h\vec{\omega}_{r/e} + \vec{\omega}_{h/r} \times \vec{\alpha}_{r/e} = D_h\vec{\omega}_{r/e} - \vec{\omega}_{r/h} \times \vec{\alpha}_{r/e} \);
\[
\vec{\alpha}_{r/e}^{(h)} = \vec{\omega}_{r/e}^{(h)} - \vec{\omega}_{3}^{(h)}\vec{\alpha}_{r/e}^{(h)},
\]
\[
\vec{\alpha}_{r/e}^{(h)} = \vec{u}_{1}(\dot{\phi} - \psi s\theta + \cdots) + \vec{u}_{2}(\dot{\theta}c\phi + \psi s\phi c\theta + \cdots) + \vec{u}_{3}(\dot{\theta}s\phi + \psi c\phi c\theta + \cdots)
\]
\[
+ \vec{u}_{4}
\dot{\theta}(\dot{\phi}c\phi + \psi s\phi c\theta) - \vec{u}_{2}\dot{\theta}(\dot{\phi} - \psi s\theta).
\]

**PROBLEM 3**

A reference frame \( \mathcal{F} \), which is rotating about its fixed origin \( O \), coincides consecutively with the reference frames \( \mathcal{F}_0 \), \( \mathcal{F}_1 \), and \( \mathcal{F}_2 \) according to the following sequence:

\[
\mathcal{F}_0 \xrightarrow{\vec{u}_{1}^{(0)}, \theta_1} \mathcal{F}_1 \xrightarrow{\vec{u}_{2}^{(1)}, \theta_2} \mathcal{F}_2
\]

Let \( P \) be a point with constant coordinates \( x_1, x_2, x_3 \) in the rotating frame \( \mathcal{F} \).

a) Find the coordinates of \( P \) in \( \mathcal{F}_1 \) and \( \mathcal{F}_0 \) when \( \mathcal{F} \) coincides with \( \mathcal{F}_2 \).

b) Find the coordinates of \( P \) in \( \mathcal{F}_2 \) while \( \mathcal{F} \) is yet coincident with \( \mathcal{F}_1 \).

c) As \( \mathcal{F} \) rotates all the way from \( \mathcal{F}_0 \) to \( \mathcal{F}_2 \), the observers in \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) use the rotation matrices \( \hat{R}_{02}^{(0)} \) and \( \hat{R}_{02}^{(1)} \) respectively in order to relate the initial and final coordinates of \( P \) that they observe. Express \( \hat{R}_{02}^{(0)} \) and \( \hat{R}_{02}^{(1)} \) using exponential rotation matrices.

**SOLUTION**

a) \[
\hat{r}_{p_2}^{(1)} = \hat{C}^{(1,2)}\hat{r}_{p_2}^{(2)}, \quad \hat{r}_{p_2}^{(0)} = \hat{C}^{(0,2)}\hat{r}_{p_2}^{(2)} = \hat{C}^{(0,1)}\hat{r}_{p_2}^{(1)}
\]
\[
\hat{r}_{p_2}^{(2)} = \hat{r}_{l_0}^{(0)} = \vec{u}_{1}x_{1} + \vec{u}_{2}x_{2} + \vec{u}_{3}x_{3}
\]
\[
\hat{C}^{(0,1)} = e^{\vec{u}_{1}\theta_{1}}, \quad \hat{C}^{(1,2)} = e^{\vec{u}_{2}\theta_{2}}, \quad \hat{C}^{(0,2)} = e^{\vec{u}_{3}\theta_{3}}
\]
\[
\hat{r}_{p_2}^{(1)} = e^{\vec{u}_{2}\theta_{2}}(\vec{u}_{1}x_{1} + \vec{u}_{2}x_{2} + \vec{u}_{3}x_{3}) = e^{\vec{u}_{2}\theta_{2}}\vec{u}_{1}x_{1} + \vec{u}_{2}x_{2} + e^{\vec{u}_{2}\theta_{2}}\vec{u}_{3}x_{3}
\]
\[
\hat{r}_{p_2}^{(1)} = (\vec{u}_{1}c\theta_{2} - \vec{u}_{3}s\theta_{2})x_{1} + \vec{u}_{2}x_{2} + (\vec{u}_{3}c\theta_{2} + \vec{u}_{1}s\theta_{2})x_{3}
\]
\[
\hat{r}_{p_2}^{(1)} = \vec{u}_{1}(x_{1}c\theta_{2} + x_{3}s\theta_{2}) + \vec{u}_{2}x_{2} + \vec{u}_{3}(x_{3}c\theta_{2} - x_{1}s\theta_{2})
\]

M. K. Özgören
\[ \mathbf{r}^{(0)}_{P_2} = e^{\tilde{u}_1(\theta_1)} [\bar{u}_1(x_1 \theta_2 + x_3 s \theta_2) + \bar{u}_2 x_2 + \bar{u}_3(x_3 c \theta_2 - x_1 s \theta_2)] \]

\[ \mathbf{r}^{(0)}_{P_2} = \bar{u}_1(x_1 \theta_2 + x_3 s \theta_2) + e^{\tilde{u}_1(\theta_2)} \bar{u}_2 x_2 + e^{\tilde{u}_1(\theta_2)} \bar{u}_3(x_3 c \theta_2 - x_1 s \theta_2) \]

\[ \mathbf{r}^{(0)}_{P_2} = \bar{u}_1(x_1 \theta_2 + x_3 s \theta_2) + (\bar{u}_2 c \theta_1 + \bar{u}_3 s \theta_1) x_2 + (\bar{u}_3 c \theta_1 - \bar{u}_2 s \theta_1)(x_3 c \theta_2 - x_1 s \theta_2) \]

\[ \mathbf{r}^{(0)}_{P_2} = \bar{u}_1(x_1 \theta_2 + x_3 s \theta_2) + \bar{u}_2(x_2 c \theta_1 + x_1 s \theta_2 \theta_2 - x_3 s \theta_1 c \theta_2) \]

\[ + \bar{u}_3(x_2 s \theta_1 - x_1 c \theta_1 s \theta_2 + x_3 c \theta_1 c \theta_2) \]

b) \[ \mathbf{r}^{(1)}_{P_1} = \hat{\mathbf{r}}^{(2)}_{P_1} = \hat{\mathbf{r}}^{(1)}_{P_1} \]

\[ \mathbf{r}^{(1)}_{P_1} = \mathbf{r}^{(0)}_{P_0} = \bar{u}_1 x_1 + \bar{u}_2 x_2 + \bar{u}_3 x_3, \quad \hat{\mathbf{r}}^{(2.1)}_{P_1} = e^{-\tilde{u}_2 \theta_2} \]

\[ \mathbf{r}^{(1)}_{P_1} = e^{-\tilde{u}_2 \theta_2}(\bar{u}_1 x_1 + \bar{u}_2 x_2 + \bar{u}_3 x_3) = e^{-\tilde{u}_2 \theta_2} \bar{u}_1 x_1 + \bar{u}_2 x_2 + e^{-\tilde{u}_2 \theta_2} \bar{u}_3 x_3 \]

\[ \mathbf{r}^{(1)}_{P_1} = (\bar{u}_1 c \theta_2 + \bar{u}_3 s \theta_2) x_1 + \bar{u}_2 x_2 + (\bar{u}_3 c \theta_2 - \bar{u}_1 s \theta_2) x_3 \]

\[ \mathbf{r}^{(1)}_{P_1} = \bar{u}_1(x_1 c \theta_2 - x_3 s \theta_2) + \bar{u}_2 x_2 + \bar{u}_3(x_1 s \theta_2 + x_3 c \theta_2) \]

c) \[ \mathbf{R}^{(0)}_{02} = \hat{\mathbf{r}}^{(2.1)}_{P_1} e^{\tilde{u}_2 \theta_2} \]

\[ \mathbf{R}^{(0)}_{02} = \hat{\mathbf{r}}^{(1)}_{P_1} \hat{\mathbf{r}}^{(0)}_{02} = \hat{\mathbf{r}}^{(1,0)}_{02} = (e^{-\tilde{u}_1 \theta_1})(e^{\tilde{u}_1 \theta_1} e^{\tilde{u}_2 \theta_2})(e^{\tilde{u}_1 \theta_1}) \]

\[ \mathbf{R}^{(1)}_{02} = e^{\tilde{u}_2 \theta_2} e^{\tilde{u}_1 \theta_1} \]

PROBLEM 4
Consider the four-link spatial mechanism shown in the figure. Joint-12 is revolute with the joint variable \( \theta_{12} \), joint-23 is cylindrical with the joint variables \( \theta_{23} \) and \( s_{23} = AB \), joint-14 is prismatic with the joint variable \( s_{14} = OC \), and joint-34 is spherical. At joint-34, the motion of link-4 relative to link-3 is described by three joint variables \( \phi_{34} \), \( \theta_{34} \), and \( \psi_{34} \), which are the Euler angles of a suitable sequence. The link parameters of the mechanism are \( b_1 = OA \), \( b_3 = BC \), and \( \beta_{14} \) is angle from \( \vec{u}_1^{(1)} \) to \( \vec{u}_1^{(4)} \) about \( \vec{u}_3^{(1)} \). Furthermore, angle(\( ABC \)) = 90° and \( AB \) is perpendicular to \( \vec{u}_1^{(1)} \).

a) Using the point-to-point loop closure equation

\[
\vec{r}_{OA} + \vec{r}_{AB} + \vec{r}_{BC} = \vec{r}_{OC}
\]

written for the joint center locations, obtain formulas to find the joint variables \( \theta_{23} \), \( s_{23} \), and \( s_{14} \) for a given value of the input joint variable \( \theta_{12} \). Indicate the closure alternatives clearly.

Suggestion: You may find it more convenient to formulate the solution as indicated below:

\[
s_{14} = f_{14}(\theta_{12}, \sigma), \quad \sigma = \pm 1; \quad s_{23} = f_{23}(\theta_{12}, s_{14}), \quad \theta_{23} = f'_{23}(\theta_{12}, s_{14}).
\]

b) Then, using the orientational loop closure equation

\[
\hat{C}(1,2) \hat{C}(2,3) \hat{C}(3,4) = \hat{C}(1,4)
\]

written for the link orientations, determine the remaining joint variables \( \phi_{34} \), \( \theta_{34} \), and \( \psi_{34} \) by choosing the most suitable sequence that gives them in the simplest possible way in terms of \( \theta_{12}, \beta_{14}, \) and already determined \( \theta_{23} \).

c) In particular, find all the non-input joint variables corresponding to each closure of the mechanism for the following numerical values:

\[
b_1 = 0.5 \text{ m}, \quad b_3 = 0.75 \text{ m}, \quad \beta_{14} = 120^\circ, \quad \theta_{12} = 30^\circ.
\]

**SOLUTION**

a) Point-to-Point Loop Closure Equation:

\[
\vec{r}_{OA} + \vec{r}_{AB} + \vec{r}_{BC} = \vec{r}_{OC}.
\]

\[
- b_1 \vec{u}_1^{(1)} + s_{23} \vec{u}_3^{(2)} + b_3 \vec{u}_1^{(3)} = -s_{14} \vec{u}_1^{(4)},
\]

\[
- b_1 \vec{u}_1^{(1/1)} + s_{23} \vec{u}_3^{(2/1)} + b_3 \vec{u}_1^{(3/1)} = -s_{14} \vec{u}_1^{(4/1)},
\]

\[
- b_1 \vec{u}_1 + s_{23} \hat{C}(1,2) \vec{u}_3 + b_3 \hat{C}(1,3) \vec{u}_1 = -s_{14} \hat{C}(1,4) \vec{u}_1.
\]

On the other hand,

\[
\hat{C}(1,2) = e^{\vec{u}_1 \theta_{12}}, \quad \hat{C}(1,3) = e^{\vec{u}_1 \theta_{12}} e^{\vec{u}_3 \theta_{23}}, \quad \hat{C}(1,4) = e^{\vec{u}_3 \beta_{14}}.
\]

Hence,

\[
- b_1 \vec{u}_1 + s_{23} e^{\vec{u}_1 \theta_{12}} \vec{u}_3 + b_3 e^{\vec{u}_1 \theta_{12}} e^{\vec{u}_3 \theta_{23}} \vec{u}_1 = -s_{14} e^{\vec{u}_3 \beta_{14}} \vec{u}_1,
\]

\[
- b_1 \vec{u}_1 + s_{23} \vec{u}_3 + b_3 e^{\vec{u}_3 \theta_{23}} \vec{u}_1 = -s_{14} e^{-\vec{u}_1 \theta_{12}} (\vec{u}_1 c \beta_{14} + \vec{u}_2 s \beta_{14}),
\]

\[
- b_1 \vec{u}_1 + s_{23} \vec{u}_3 + b_3 (\vec{u}_1 c \theta_{23} + \vec{u}_2 s \theta_{23}) = -s_{14} (\vec{u}_1 c \beta_{14} + \vec{u}_2 s \beta_{14} c \theta_{12} - \vec{u}_3 s \beta_{14} s \theta_{12}).
\]

M. K. Ö zgören
The corresponding scalar equations are

\[
b_3 c \theta_{23} = b_1 - s_{14} c \beta_{14}, \quad \text{Eq. (i)}
\]
\[
b_3 s \theta_{23} = -s_{14} s \beta_{14} c \theta_{12}, \quad \text{Eq. (ii)}
\]
\[
s_{23} = s_{14} s \beta_{14} s \theta_{12}. \quad \text{Eq. (iii)}
\]

By adding squares of Eqs. (i) and (ii), we get

\[
(c^2 \beta_{14} + s^2 \beta_{14} c^2 \theta_{12}) s_{14}^2 - 2(b_1 c \beta_{14}) s_{14} - (b_2^2 - b_1^2) = 0.
\]

Two possible solutions to this equation are

\[
s_{14} = \frac{b_1 c \beta_{14} + \sigma \sqrt{b_2^2 c^2 \beta_{14} + (b_3^2 - b_1^2)s^2 \beta_{14} c^2 \theta_{12}}}{c^2 \beta_{14} + s^2 \beta_{14} c^2 \theta_{12}}; \quad \sigma = \pm 1.
\]

With this solution, Eq. (iii) gives \( s_{23} \) readily as

\[
s_{23} = s_{14} s \beta_{14} s \theta_{12}.
\]

Finally, from the ratio of Eqs. (i) and (ii), we get

\[
\theta_{23} = \tan^{-1}(-s_{14} s \beta_{14} c \theta_{12} : b_1 - s_{14} c \beta_{14}).
\]

Here, \( \sigma \) is the closure indicator. Once, its value (+1 or -1) is selected, the closure will be defined and the corresponding variables \( (s_{14}, s_{23}, \text{and } \theta_{23}) \) will be uniquely determined by the preceding expressions.

b)

Orientational Loop Closure Equation:

\[
\hat{C}^{(1,2)} \hat{C}^{(2,3)} \hat{C}^{(3,4)} = \hat{C}^{(1,4)}
\]

\[
\hat{C}^{(1,2)} = e^{\hat{u}_1 \theta_{12}}, \quad \hat{C}^{(2,3)} = e^{\hat{u}_2 \theta_{23}}, \quad \hat{C}^{(3,4)} = ? , \quad \hat{C}^{(1,4)} = e^{\hat{u}_3 \beta_{14}}.
\]

The orientational equation leads to

\[
\hat{C}^{(3,4)} = e^{-\hat{u}_3 \theta_{23}} e^{-\hat{u}_1 \theta_{12}} e^{\hat{u}_3 \beta_{14}}.
\]

This equation implies that the most suitable sequence for the spherical joint (Joint-34) is 3-1-3. That is,

\[
\hat{C}^{(3,4)} = e^{\hat{u}_3 \phi_{34}} e^{\hat{u}_1 \theta_{34}} e^{\hat{u}_3 \psi_{34}} = e^{-\hat{u}_3 \theta_{23}} e^{-\hat{u}_1 \theta_{12}} e^{\hat{u}_3 \beta_{14}}.
\]

Hence, by direct comparison, the associated joint variables are determined simply as

\[
\phi_{34} = -\theta_{23}, \quad \theta_{34} = -\theta_{12}, \quad \psi_{34} = \beta_{14}.
\]
c)
With the given numerical values, we get the following solutions for each closure:

First closure with $\sigma = +1$:

$s_{14} = 0.3846 \text{ m}, \quad s_{23} = 0.1665 \text{ m}, \quad \theta_{23} = -22.62^\circ; \\
\phi_{34} = 22.62^\circ, \quad \theta_{34} = -30^\circ, \quad \psi_{34} = 120^\circ.$

Second closure with $\sigma = -1$:

$s_{14} = -1.000 \text{ m}, \quad s_{23} = -0.433 \text{ m}, \quad \theta_{23} = 90^\circ; \\
\phi_{34} = -90^\circ, \quad \theta_{34} = -30^\circ, \quad \psi_{34} = 120^\circ.$