

Let \mathbb{T} be a connected groupoid.

Let $F: \mathbb{T} \rightarrow \text{SET}$ be a functor. Then for any $b \in \mathbb{T}$ $F(b)$ is a $\Pi_b = \Pi(b, b)$ -set.

If $\alpha \in \Pi(b, c)$ is a morphism from b to c , then $F(\alpha): F(b) \rightarrow F(c)$ is a bijection.

If $G: \mathbb{T} \rightarrow \text{SET}$ is another functor and φ is a natural transformation then

$$\begin{array}{ccc} F(b) & \xrightarrow{F(\alpha)} & F(c) \\ \varphi(b) \downarrow & & \downarrow \varphi(c) \\ G(b) & \xrightarrow{G(\alpha)} & G(c) \end{array} \text{ is a commutative diagram.}$$

In particular, if $\alpha \in \Pi_b = \Pi(b, b)$, then

$$\begin{array}{ccc} F(b) & \xrightarrow{F(\alpha)} & F(b) \\ \varphi(b) \downarrow & & \downarrow \varphi(b) \\ G(b) & \xrightarrow{G(\alpha)} & G(b) \end{array} \text{ is also commutative.}$$

Hence, the map $\varphi(b): F(b) \rightarrow G(b)$ is Π_b -equivariant.

So we have a functor $\Sigma_b: [\mathbb{T}, \text{SET}] \rightarrow \Pi_b\text{-SET}$, where

$\Pi_b\text{-SET}$ is the category of $\Pi_b\text{-SET}$'s with morphisms equivariant maps.

Now define a functor κ_b in the opposite direction: Fix some $b \in \mathbb{T}$ and let A be a Π_b -set.

$\Phi(A): \mathbb{T} \rightarrow \text{SET}$ is defined as follows:

For any $x \in B$, let $\widehat{\Phi}(A)(x) = A \times_{\pi} \Pi(b, x)$, where

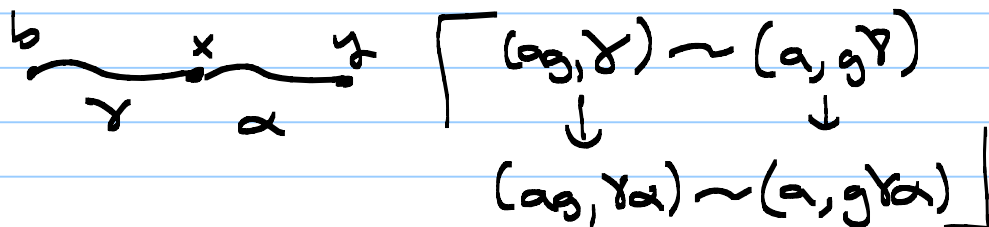
$A \times_{\pi} \Pi(b, x)$ is $A \times \Pi(b, x)$ modulo the relation

$(ag, \gamma) \sim (a, g\gamma)$, for any $(a, g, \gamma) \in A \times \Pi \times \Pi(b, x)$

For any $\alpha \in \Pi(x, y)$ morphism in Π let

$\widehat{\Phi}(A)(\alpha): A \times_{\pi} \Pi(b, x) \rightarrow A \times_{\pi} \Pi(b, y)$

$[ag, \gamma] \mapsto [ag, \gamma\alpha]$



Examples:

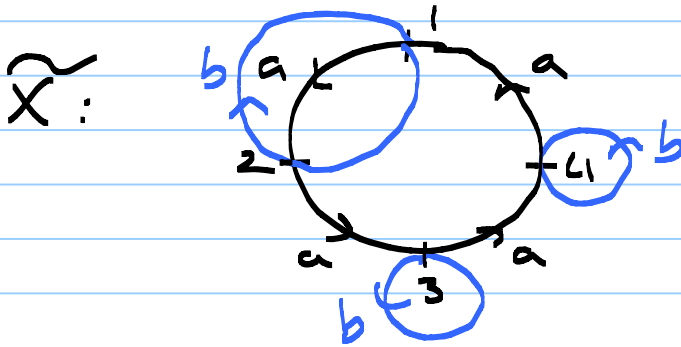
1) k -fold covers $X = \sqrt[n]{S^1}$. Since $\pi_1(X) = F_n$ any k -fold cover is determined by an action on F_n on $\{1, 2, \dots, k\}$ i.e. a homomorphism

$$F_n \rightarrow S_k.$$

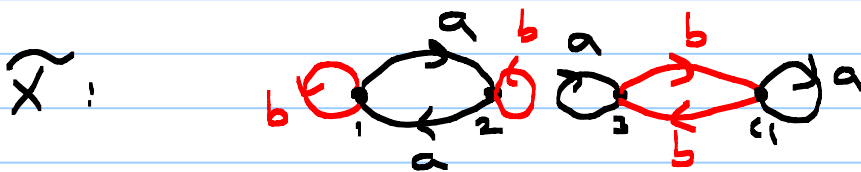
For example, let $n=2$ and $k=4$.

$$F_2 = \langle a, b \rangle, \quad X: \text{figure-eight}$$

$$\text{i) } F_2 \rightarrow S_4, \quad a \mapsto (1234), \quad b \mapsto (12)$$

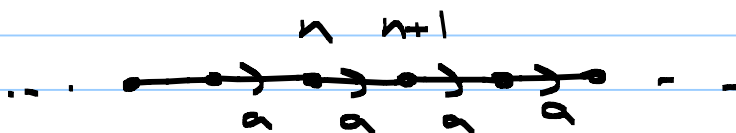


$$\text{ii) } F_2 \rightarrow S_4, \quad a \mapsto (12), \quad b \mapsto (34)$$

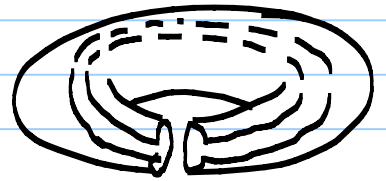
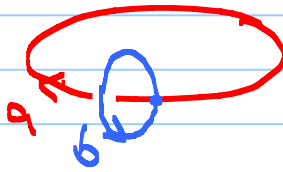
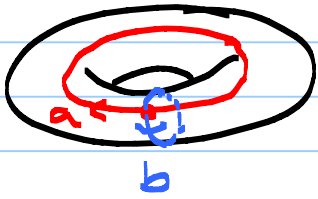


2) $X = S^1$, $\pi_1(X) = \mathbb{Z}$, \mathbb{Z} acts on \mathbb{Z} as

$$n \cdot k = n+k.$$



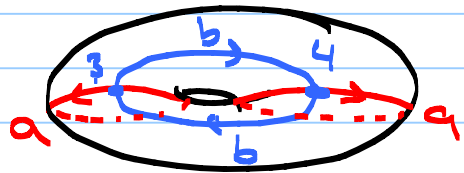
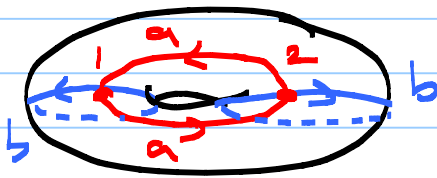
3) $X = T^2 = S^1 \times S^1$, $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}$



i) 4-fold cover of T^2

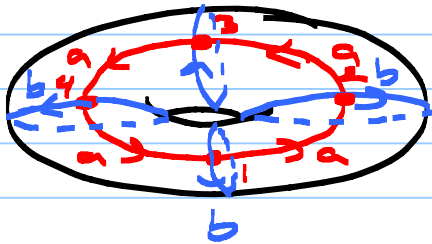
$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z} \rightarrow S_4$, $a \mapsto (12)$, $b \mapsto (34)$

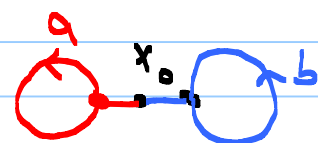
\tilde{X} :



ii) $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z} \rightarrow S_4$ $a \mapsto (1234)$, $b \mapsto e$

\tilde{X}

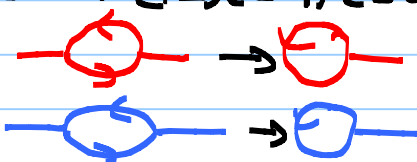


4) A normal S_3 -covering of $X =$ 

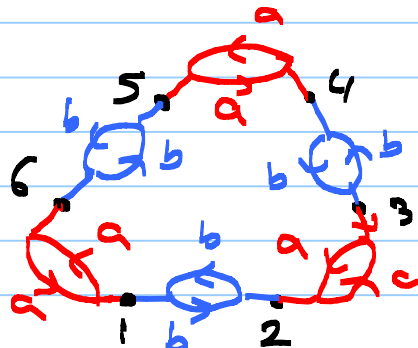
$\pi_1(X) \cong F_2 = \langle a, b \rangle$ let the action be given by

$a \mapsto (16)(23)(45)$

$b \mapsto (12)(34)(56)$



} double covers



$\tilde{X} \rightarrow X$
6-fold regular covering.

Fact: For a transport simple space B the equivalence $\text{COV}_B \rightarrow \Pi_b\text{-SET}$ show that every covering is determined by an action of $\Pi_b = \Pi_1(B, b)$ on some set. Connected covers corresponds to transitive actions.

Finally, it is clear that every transitive action is of the form

$$G \times X \rightarrow X, \text{ where } X = \{Hg \mid g \in G\} \text{ for some}$$

subgroup $H \leq G$. Moreover, such actions, up to isomorphism, are determined by the conjugacy class of H in G .

Any transitive action of a group G is determined by the conjugacy class of the stabilizer of any element:

$$G \times X \rightarrow X, \quad H = \text{Stab}_G(x).$$

Moreover, in this case we may take $X = \{Hg \mid g \in G\}$ the set of all right cosets of H and the action of G on X is just right multiplication.

Note that for any points $x, y \in X$ if $y = gx$ then

$$\text{Stab}_G(y) = g \text{Stab}_G(x) g^{-1}$$

For a connected covering $P: E \rightarrow B$, fix some $x \in E$, $b = p(x)$. Then this cover is uniquely determined by the action of $G = \Pi_1(B, b)$ on the fiber $F_x = p^{-1}(b)$, namely the $\text{Stab}_G(x) = H = P_x \Pi_1(E, x)$.

The equivalence of categories Triples that any automorphism of $P: \mathcal{E} \rightarrow \mathcal{B}$ (i.e. a double transformation) is an automorphism of the group action $G \times X \rightarrow X$.

It is easy to see that automorphism group is NH/H , where N is the normalizer of H in G (note that as in the case of coverings, any automorphism is determined by its action on a single element).

Automorphism of Group Actions

Let $\varphi: G \times X \rightarrow X$ be a transitive action. Let

$H = \text{Stab}_G(x_0)$ of some point $x_0 \in X$. Then

$\text{Aut}(\varphi) \cong N/H$, where N is the normalizer of H in G .

Proof: Let $\phi: X \rightarrow X$ be an automorphism

of φ . Then $\varphi(g, \phi(x)) = \phi(\varphi(g, x))$, for all $(g, x) \in G \times X$.

In particular, $g \phi(x_0) = \phi(g x_0)$, for all $g \in G$.

Let $g \in H = \text{Stab}_G(x_0)$. Then $g \phi(x_0) = \phi(g x_0) = \phi(x_0)$

so that $g \in \text{Stab}_G(\phi(x_0)) = g, H \bar{g}^{-1}$, where

$\phi(x_0) = g, x_0$. This shows that $H \subseteq g, H \bar{g}^{-1}$.

Similarly, if $g \in \text{Stab}_G(\phi(x_1)) = g, H \bar{g}^{-1}$, then $g \phi(x_0) = \phi(x_1)$.

So, $\phi(x_0) = g\phi(x_0) = \phi(gx_0)$ and hence $x_0 = gx_0$ because ϕ is a bijection. So, $g \in \text{Stab}_G(x_0) = H$.

Hence, $H = gHg^{-1}$ or equivalently $g \in N_G(H)$.

Let $X = \{Hg \mid g \in G\}$ be the set of right cosets of H , and $\varphi(g_1, Hg_2) = Hg_2g_1$, for any $g_1, g_2 \in G$, the usual action.

For any $g \in N_G(H)$, define $\phi_g: X \rightarrow X$, $Hg_1 \mapsto \bar{g}Hg_1$.

Claim: ϕ_g is an automorphism of the action φ .

$$\begin{aligned}\varphi(g_2, \phi_g(Hg_1)) &= \varphi(g_2, \bar{g}Hg_1) \\ &= \bar{g}Hg_1g_2 \\ &= \bar{g}\varphi(g_2, Hg_1) \\ &= \phi_g(\varphi(g_2, Hg_1)).\end{aligned}$$

(Of course, one has to first notice that ϕ_g is a well-defined map of X to itself, because

$$\phi_g(Hg_1) = \bar{g}Hg_1 = H\bar{g}g_1 \in X$$

is a right coset, for any $g_1 \in G$ and $g \in N$.

The above arguments prove that

$$\text{Aut}(\varphi) \cong N/H = N_G(H)/H.$$

Monodromy and Regular Coverings:

Let $P: \tilde{X} \rightarrow X$ be a (connected) covering map. Fix some $x_0 \in X$ and let $F_{x_0} = P^{-1}(x_0)$ be the fiber over x_0 .

For any choice $\tilde{x}_0 \in F_{x_0}$, consider the monodromy $\varphi_{\tilde{x}_0}: \pi_1(X, x_0) \rightarrow F_{x_0}$, given by $\varphi([\gamma]) = \tilde{\gamma}(1)$, for any $[\gamma] \in \pi_1(X, x_0)$, where $\tilde{\gamma}$ is the unique lift of γ starting at \tilde{x}_0 .

This defines a homomorphism

$$\begin{aligned} \varphi: \pi_1(X, x_0) &\rightarrow \text{Aut}(F_{x_0}) = \text{Per}(F_{x_0}) \\ \gamma &\longmapsto (\varphi_{\tilde{x}_0}[\gamma])_{\tilde{x}_0 \in F_{x_0}} \end{aligned}$$

whose kernel is $\bigcap_{g \in \pi_1(X, x_0)} H^g$. Hence, it follows that

if the covering is normal then $H \triangleleft \pi_1(X, x_0)$ and hence $\ker \varphi = H$.

Clearly, the converse is also true: if $\ker \varphi = H$ then the covering is normal.

The arguments on page 9 show that

$$\text{Aut}(P: \tilde{X} \rightarrow X) = N/H, \text{ when } N = N_{\mathcal{O}}(H).$$

In particular, if the covering is normal then

$$\text{Aut}(P: \tilde{X} \rightarrow X) \cong \pi_1(X, x_0)/H \cong \text{Im} \varphi, \text{ where } \varphi \text{ is the monodromy homomorphism.}$$

Note that there is also a natural injective map $\Psi: \text{Aut}(P: \tilde{X} \rightarrow X) \rightarrow \text{Per}(F_{X_0})$. For a general covering $\text{Im} \Psi \cong \mathcal{N}H/H \subseteq \text{Per}(F_{X_0})$ and $\text{Im} \Psi \cong \pi_1(X, x_0) / \bigcap_{y \in \pi_1(X, x_0)} H^y$ and thus, in general,

$\text{Im} \Psi$ is "bigger" than $\text{Im} \Psi \cong \text{Aut}(P: \tilde{X} \rightarrow X)$.

If the covering is normal then $\mathcal{N}H = \pi_1(X, x_0)$ and $\bigcap_{y \in \pi_1(X, x_0)} H^y = H$ so that $\text{Im} \Psi \cong \text{Im} \Psi$.

However, even in this case $\text{Im} \Psi$ and $\text{Per} \Psi$ need not to be the same.

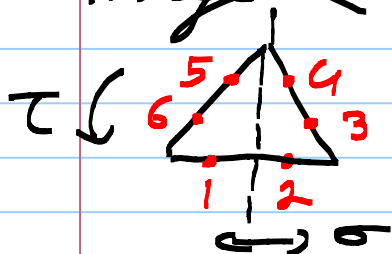
Example: The Example 4 exhibits the 6-fold regular cover $P: \tilde{X} \rightarrow X$, where $\pi_1(X, x_0) \cong F_2$ and $\text{Aut}(P) \cong S_3$.

Recall that $\varphi: \pi_1(X, x_0) \rightarrow S_6$ has image $\langle \varphi(a), \varphi(b) \rangle \cong S_3 \leq S_6$, where

$\varphi(a) = (16)(23)(45)$ and $\varphi(b) = (12)(34)(56)$.

On the other hand, $\psi: \text{Aut}(P) \rightarrow S_6$ has

image $\langle \psi(\tau), \psi(\sigma) \rangle = \langle (135)(246), (12)(36)(45) \rangle$.



The images $\varphi(a)$ and $\varphi(b)$ are both non-geometric and thus $\text{Im}(\varphi)$ and $\text{Im} \psi$ are not the same subgroups.

Here T is the $2\pi/3$ -radian counterclockwise rotation and σ is the reflection with respect to the y -axis.

Order Automorphisms of F_n .

Let K be a finite group of order n , generated by r elements. Then there is an exact sequence of groups:

$$0 \rightarrow F_s \rightarrow F_n \rightarrow K \rightarrow 0, \text{ where}$$

$$s-1 = k(r-1) \text{ and hence } s = 1 + k(r-1).$$

Example: If $r=2$ and $K = D_{2n}$, the dihedral group of order $2n$, then $s = 1 + 2n$.

Claim: The canonical map $K \rightarrow \text{Aut}(F_s) \rightarrow \text{Out}(F_s)$

is injective provided that $r \geq 2$.

Proof: If some $g \in G$ acts trivial on F_s (by conjugation) then $g \in C_{F_r}(F_s)$, the centralizer of F_s in F_r .

In a free group two elements commute if and only if they are powers of a common element. This implies that F_s is cyclic, which is not possible if $r \geq 2$.
This finishes the proof. \square

Example: Both Z_6 and $D_6 = S_3$ are groups of order 6 generated by two elements. Hence, $\text{Out}(F_2)$ contains Z_6 and S_3 as subgroups.

Thanks: I would like to thank my dear colleague Gökhan Benli for explaining the proof of the above claim.

