# STAT 361

# HOMEWORK 4

# SOLUTIONS

## PART I :

a) Newton - Raphson Method :

$$x_{(i+1)} = x_{(i)} - \frac{f(x_{(i)})}{f'(x_{(i)})}$$

where $f(x) = x + \sin(x) - \frac{x^3}{3}$

$$f'(x) = 1 + \cos(x) - \frac{\cancel{3}x^2}{\cancel{3}}$$

$$\boxed{x_{(i+1)} = x_{(i)} - \frac{x_{(i)} + \sin(x_{(i)}) - \frac{x_{(i)}^3}{3}}{1 + \cos(x_{(i)}) - x_{(i)}^2}}$$

b) In order to find a starting point in Newton-raphson method, sketching the plot of this function is a good approach. Initial point is very important in Newton-raphson method in order to approach the root of the function correctly and fast. When initial point is chosen too far from the root, then we may not end up with the correct solution. So in this question we should plot the function and choose an initial point close to the point where the function intersects the line $y=0$.

c) In this question the function f intersects the line $y=0$ at three distinct points, which means that this function has three real roots. In order to find each root with Newton-Raphson method, we should change the initial value at each case. In other words, we should apply Newton-Raphson algorithm 3-times, and each time we should choose a different initial point close to the root which we aim to find.

## PART II :

a) Secant Method :

$$x_{(i+1)} = \frac{x_{(i-1)} f(x_{(i)}) - x_{(i)} f(x_{(i-1)})}{f(x_{(i)}) - f(x_{(i-1)})}$$

where $f(x) = x + \sin(x) - \left(\frac{x^3}{3}\right)$

$$x_{(i+1)} = \frac{x_{(i-1)}\left(x_{(i)} + \sin(x_{(i)}) - \left(x_{(i)}^3/3\right)\right) - x_{(i)}\left(x_{(i-1)} + \sin(x_{(i-1)}) - \left(x_{(i-1)}^3/3\right)\right)}{x_{(i)} + \sin(x_{(i)}) - \left(x_{(i)}^3/3\right) - \left(x_{(i-1)} + \sin(x_{(i-1)}) - \left(x_{(i-1)}^3/3\right)\right)}$$

$$x_{(i+1)} = \frac{x_{(i-1)} \sin(x_{(i)}) - x_{(i)} \sin(x_{(i-1)}) + \left(x_{(i)} x_{(i-1)}/3\right)\left(x_{(i-1)}^2 - x_{(i)}^2\right)}{\left(x_{(i)} - x_{(i-1)}\right) + \left(\sin(x_{(i)}) - \sin(x_{(i-1)})\right) + \left(\frac{x_{(i-1)}^3 - x_{(i)}^3}{3}\right)}$$

b) Like in the Newton-Raphson method, determination of initial points in Secant method is also very important in finding the root(s) of the function. Again, sketching the plot of function is a good method for determining initial points effectively. By looking at the plot we can choose 2 points that are close to the root of the function.

c) Since this function has 3 roots, we can find these roots by applying Secant method 3 times with different initial points at each case.

## PART III:

a) Fixed point iteration method:

$$f(x) = x + \sin(x) - (x^3/3) = 0$$

$$x = (x^3/3) - \sin(x) = g_1(x)$$

for $x = 1$ $\quad g_1(1) = -0.5081 \notin [1,3]$ $\quad g_1(x)$ does not have a fixed point in this interval.

$$x = \sqrt[3]{3(x + \sin(x))} = g_2(x)$$

for $x=1$ $\quad g_2(1) = 1.7678 \in [1,3]$ ✓

$x=3$ $\quad g_2(3) = 2.1122 \in [1,3]$ ✓

$x=2$ $\quad g_2(2) = 2.0589 \in [1,3]$ ✓

$g_2(x)$ is an increasing function in $[1,3]$

for $x \in [1,3]$

$g_2(x) \in [1,3]$

Let us check <u>Cond 1:</u>

$g_2(x)$ is a continuous func. in $[1,3]$ ✓
and

$|g_2'(x)| \leq L$ with $L < 1$ for all $x \in [1,3]$

$$g_2'(x) = \frac{1}{\cancel{3}}\left(3(x+\sin(x))\right)^{-2/3}\left(\cancel{3}(1+\cos(x))\right)$$

when we plot $|g_2'(x)|$ in R, we can see that $|g_2'(x)| \leq 0.5$ for $x \in [1,3]$

So Cond. 1 is satisfied and we can say that $g_2(x)$ has a fixed point in $[1,3]$, and it converges to the unique root of $f(x)=0$ in $[1,3]$ for any $x_0$ in $[1,3]$.

$$x_{(i+1)} = g_2(x_{(i)})$$

$$x_{(i+1)} = \left[3\left(x_{(i)}+\sin(x_{(i)})\right)\right]^{1/3}$$