STAT 361

HOMEWORK 3

Solution

(1) $$\theta = \int_{-1.96}^{1.96} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

In order to use Monte Carlo Integration method, we should make a transformation such that the limits of the integral are become 0 to 1.

$$y = \frac{x-(-1.96)}{1.96-(-1.96)} = \frac{x+1.96}{2*1.96} \qquad \begin{matrix} x=-1.96 \\ y=0 \end{matrix} \qquad \begin{matrix} x=1.96 \\ y=1 \end{matrix}$$

$$x = 2*1.96y - 1.96$$
$$dx = 2*1.96\, dy$$

$$\theta = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-(2*1.96y-1.96)^2/2}\, 2*1.96\, dy$$

$$\theta = \frac{2*1.96}{\sqrt{2\pi}} \int_0^1 \underbrace{e^{-1.96^2(2y-1)^2/2}}_{h(y)}\, dy \rightarrow \text{For } u \sim U(0,1) \quad \int_0^1 h(u)\, du = E(h(u))$$

$$\hat{\theta} = \frac{2*1.96}{\sqrt{2\pi}} \left[\frac{1}{n} \sum_{i=1}^{n} h(u_i) \right] = \frac{2*1.96}{\sqrt{2\pi}} \left[\frac{1}{n} \sum_{i=1}^{n} e^{-1.96^2(2u_i-1)^2/2} \right]$$

Algorithm:

(1) Generate $u_i \sim U(0,1)$ $i=1, \dots, n$ (n must be large)

(2) Calculate average of $h(u_i)$ values for $i=1, \dots, n$

(3) Multiply the average value in step 2 with $\frac{2*1.96}{\sqrt{2\pi}}$

The exact value of θ is:

$$\theta = \int_{-1.96}^{1.96} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = P(-1.96 < x < 1.96) \quad \text{where } x \sim N(0,1)$$

$= 0.475 * 2 \rightarrow$ from standard normal table

$= \underline{0.95}$

(2) a) $\sum_{i=1}^{8} X_i \sim N(550, 150^2)$

$$Y = \sum_{i=1}^{8} X_i$$

$$P(Y>600) = \int_{600}^{\infty} \frac{1}{\sqrt{2\pi}\,150} e^{-\frac{(y-550)^2}{2*150^2}}\, dy$$

Since we cannot integrate this function, let us solve it with Monte Carlo Integration method. Therefore, we should transform our variable y in order to obtain limits from 0 to 1.

$$z = \frac{600}{y} \qquad \begin{matrix} y=600 \\ z=1 \end{matrix} \qquad \begin{matrix} y=\infty \\ z=0 \end{matrix}$$

$$y = \frac{600}{z}$$

$$dy = -\frac{600}{z^2}\, dz$$

$$P(Y>600) = \cancel{-}\int_{0}^{1} \frac{1}{\sqrt{2\pi}\,150} e^{-\frac{\left(\frac{600}{z}-550\right)^2}{2*150^2}} \left(\cancel{-}\frac{600}{z^2}\right) dz$$

$$= \frac{\cancel{600}\,4}{\sqrt{2\pi}\,\cancel{150}_1} \int_{0}^{1} \underbrace{\frac{e^{-\frac{\left(\frac{600}{z}-550\right)^2}{2*150^2}}}{z^2}}_{h(z)}\, dz$$

For $u \sim U(0,1)$

$$\int_0^1 h(u)\, du = E(h(u))$$

$$\widehat{P(Y>600)} = \frac{4}{\sqrt{2\pi}}\left[\frac{1}{n}\sum_{i=1}^{n} h(u_i)\right] = \frac{4}{\sqrt{2\pi}}\left[\frac{1}{n}\sum_{i=1}^{n} \frac{e^{-\frac{\left(\frac{600}{u_i}-550\right)^2}{2*150^2}}}{u_i^2}\right]$$

Algorithm:

1. Generate $u_i \sim U(0,1)$ for $i=1,\dots,n$ (large n)
2. Calculate average of $h(u_i)$ values for $i=1,\dots,n$
3. Multiply the average value in step 2 with $\frac{4}{\sqrt{2\pi}}$

b) $P(Y>600) = \int_{600}^{\infty} \frac{1}{\sqrt{2\pi}\,150} e^{-\frac{(y-550)^2}{2*150^2}} dy = 0.3694$ is the exact value of this prob. calculated by R.