STAT 361

Homework 2

KEY

Q1)

a) $f(x)=\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$; $-\infty<x<\infty$, $-\infty<\mu<\infty$, $\sigma>0$

For $\mu=0$, $\sigma=1$

$$f(x)=\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty<x<\infty$$

$$g(x)=\frac{\lambda}{2} e^{-\lambda|x|}, \quad -\infty<x<\infty$$

For $\lambda=1$

$$g(x)=\frac{1}{2} e^{-|x|}, \quad -\infty<x<\infty$$

We should find c such that,

$$c=\sup_x \frac{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}{\frac{1}{2} e^{-|x|}} = \sup_x \frac{2}{\sqrt{2\pi}} e^{-x^2/2+|x|}$$

$\Rightarrow$ In order to maximize this func. we should find the max. value for func. $(-x^2/2+|x|)$

$$\frac{d}{dx}\left(-\frac{x^2}{2}+|x|\right)=0$$

$x>0$

$$-\frac{\cancel{2}x}{\cancel{2}}+1=0$$

$$x=1$$

$x<0$

$$-\frac{\cancel{2}x}{\cancel{2}}-1=0$$

$$x=-1$$

For $x=\mp 1$

$$-\frac{x^2}{2}+|x| = -\frac{1}{2}+1=\frac{1}{2}$$

So, $c = \frac{2}{\sqrt{2\pi}} e^{1/2}$

Now, we should find an algorithm in order to generate random number from $g(x)$, we can use inverse transform method.

$$g(x) = \frac{1}{2} e^{-|x|} \quad , \quad -\infty < x < \infty$$

$$G(x) = \int_{-\infty}^{x} \frac{1}{2} e^{-|t|} dt$$

For $x < 0$:

$$G(x) = \int_{-\infty}^{x} \frac{1}{2} e^{t} dt = \frac{1}{2} e^{t} \Big|_{-\infty}^{x} = \frac{1}{2} \left(e^{x} - \cancel{e^{-\infty}}^{\,0}\right) = \frac{e^{x}}{2}$$

For $x \geq 0$:

$$G(x) = \int_{-\infty}^{0} \frac{1}{2} e^{t} dt + \int_{0}^{x} \frac{1}{2} e^{-t} dt = \frac{1}{2} e^{t} \Big|_{-\infty}^{0} + \frac{1}{2} \left(-e^{-t}\right)\Big|_{0}^{x}$$

$$= \frac{1}{2}\left(e^{0} - \cancel{e^{-\infty}}^{\,0}\right) + \frac{1}{2}\left(-e^{-x} - (-e^{0})\right) = \frac{1}{2} + \frac{1}{2}\left(1 - e^{-x}\right) = \frac{1}{2}\left(2 - e^{-x}\right)$$

For $x < 0$:

$$G(x) = \frac{e^{x}}{2} = u \Rightarrow x = \ln(2u)$$

$$x < 0$$
$$\ln(2u) < 0$$
$$2u < e^{0}$$
$$\text{when } \boxed{u < \frac{1}{2}}$$

For $x \geq 0$:

$$G(x) = \frac{1}{2}\left(2 - e^{-x}\right) = u \Rightarrow e^{-x} = 2 - 2u$$
$$x = -\ln(2(1-u))$$

$$x \geq 0$$
$$-\ln(2(1-u)) \geq 0$$

$$\ln(2(1-u)) \leq 0$$
$$2(1-u) \leq e^{0}$$
$$1 - u \leq \frac{1}{2}$$
$$\text{when } \boxed{u \geq 1/2}$$

<u>Algorithm:</u>

① Generate $y \sim g(y)$ with the following algo.

Generate $u_1 \sim \mathcal{U}(0,1)$

if $u_1 < \frac{1}{2}$ set $y = \ln(2u_1)$

else set $y = -\ln(2(1-u_1))$

② Generate $u \sim \mathcal{U}(0,1)$

③ Check $u < \dfrac{f(y)}{\frac{2\sqrt{e}}{\sqrt{2\pi}}\, g(y)}$ if satisfied set $x = y$
else go back to step 1.

b) Acceptance prob $= \dfrac{1}{c} = \dfrac{1}{\frac{2\sqrt{e}}{\sqrt{2\pi}}} = \dfrac{\sqrt{2\pi}}{2\sqrt{e}}$