Specific methods for non-uniform #s:

1. The use of central limit thm:

Note that if $u_1, u_2, \ldots, u_n \overset{iid}{\sim} unif(0,1)$, then

$$\frac{\sum_{i=1}^{n} u_i - E\left[\sum_{i=1}^{n} u_i\right]}{\sqrt{Var\left(\sum_{i=1}^{n} u_i\right)}} \xrightarrow{d} N(0,1) \quad \text{by CLT.}$$

As $n \to \infty$, the distr. tends to normal distr. But, of course, in practice, we settle for some finite # of n. So, how large should n be? $n=2$ is not suitable, bc. the distr. has a triangular shape for $n=2$. For $n=3$, the distr. is already nicely "bell-shaped". The answer depends on how close an approximation is required.

Consider $n=12$. Then,

$$E(u_i) = \frac{1}{2} \Rightarrow E\left(\sum_{i=1}^{12} u_i\right) = \sum_{i=1}^{12} E(u_i) = 12 \cdot \frac{1}{2} = 6$$

$$V(u_i) = \frac{1}{12} \Rightarrow V\left(\sum u_i\right) \underset{\text{since indep.}}{=} \sum_{i=1}^{12} V(u_i) = 12 \cdot \frac{1}{12} = 1$$

$$\Rightarrow \sum_{i=1}^{12} u_i - 6 \xrightarrow{d} N(0,1)$$

(Note that $> \mp 6$ values do not occur, which could be a problem).

2. The Box-Muller method:

If U_1 & U_2 are two indep. $U(0,1)$ r.v.s, then Box & Muller (1958) showed that,

$$X = \sqrt{(-2\ln U_1)}\,\cos(2\pi U_2) \quad \text{and}$$

$$Y = \sqrt{(-2\ln U_1)}\,\sin(2\pi U_2) \quad \text{are indep. } N(0,1) \text{ r.v.s.}$$

Since this method requires the computation of sine & cosine functions, it is computationally time-consuming.

3. The Polar method:

A rejection method that gets around the time-consuming calculations in Box-Muller.

Note that: if $U \sim Unif(0,1)$ then

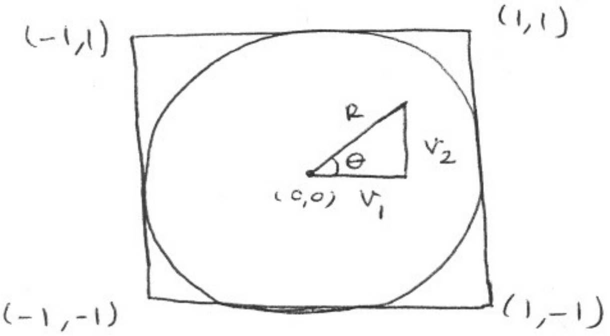
$$2U \sim Unif(0,2)$$

$$2U-1 \sim Unif(-1,1)$$

$ we generate #s U_1 & U_2, set

$V_1 = 2U_1 - 1$ & $V_2 = 2U_2 - 1$, and accept #s only if $V_1^2 + V_2^2 \le 1$.

Note that this will lead to (V_1, V_2) uniformly distributed in the circle.



$$\sin\theta = \frac{V_2}{R} = \frac{V_2}{\sqrt{V_1^2 + V_2^2}}$$

$$\cos\theta = \frac{V_1}{R} = \frac{V_1}{\sqrt{V_1^2 + V_2^2}}$$

$\Rightarrow$ Box-Muller transformation can be written as

$$X = \sqrt{-2\log U}\ \frac{V_1}{\sqrt{V_1^2+V_2^2}}$$

$$Y = \sqrt{-2\log U}\cdot \frac{V_2}{\sqrt{V_1^2+V_2^2}}$$

Since $R^2 = V_1^2+V_2^2$ is $\sim$ Unif$(0,1)$ & indep. of Θ, we can use it for U above. Let $S = R^2$.

$$\Rightarrow\ X = \sqrt{-2\ln S}\ \frac{V_1}{\sqrt{S}} = V_1\left(\frac{-2\ln S}{S}\right)^{1/2} \quad \text{and}$$

$$Y = V_2\left(-\frac{2\ln S}{S}\right)^{1/2}$$

are indep. $N(0,1)$ s.

The Algorithm is:

1. Generate U_1 & U_2.
2. Set $V_1 = 2U_1-1$, $V_2 = 2U_2-1$, $S = V_1^2+V_2^2$
3. If $S>1$, go to step 1.
4. Set $X = \left(\frac{-2\ln S}{S}\right)^{1/2}\cdot V_1$, $Y = \left(\frac{-2\ln S}{S}\right)^{1/2}\cdot V_2$

4. Generating from χ^2_m:

Recall: $\sum_{i=1}^{m} Z_i^2 \sim \chi^2_m$ where $Z_i \sim N(0,1)$.

So, to generate from χ^2_m, simply generate m std. normal variates, square them, and sum them up. But, since simulating from $N(0,1)$ might be time-consuming, the approach below might be easier:

If m is even, χ^2_m is simply Gamma$\left(\frac{m}{2}, \frac{1}{2}\right)$.
(we already talked about how to generate from Gamma by using Exp.)

If m is odd, first generate from χ^2_{m-1} by using Gamma$\left(\frac{m-1}{2}, \frac{1}{2}\right)$ and then adding Z_i^2.

To generate from Gamma when shape parameter is not integer, refs:

Cheng (1977) Applied Statistics, 26(1), 71-74.

Kinderman & Monahan (1980) New methods for generating Student's t & gamma vrs. Computing, 25, 369-377.