# STAT 361

## Homework 1

## KEY

Q1) a) $x_{i+1} = a * x_i \mod m$

$x_0 = 1$

$x_1 = 13 * 1 \mod 10 = 3$

$x_2 = 13 * 3 \mod 10 = 9$

seq: 1, 3, 9

random numbers from U(0, 1): 0.1, 0.3, 0.9

b) $F(z) = \int_{-\infty}^{z} e^{-t} e^{-e^{-t}} dt = \cancel{-}\int_{e^{-z}}^{\infty} \cancel{y} e^{-y} \left(\cancel{-}\frac{1}{\cancel{y}}\right) dy$

Let $e^{-t} = y$, the boundaries:

$t = -\ln y$    $t = -\infty$, $y = \infty$

$dt = -\frac{1}{y} dy$    $t = z$, $y = e^{-z}$

$$F(z) = \int_{e^{-z}}^{\infty} e^{-y} dy = -e^{-y}\Big|_{e^{-z}}^{\infty} = -0 - \left(-e^{-e^{-z}}\right) = e^{-e^{-z}}$$

$F(z) = e^{-e^{-z}}$ For the inverse transform method $e^{-e^{-z}} = u$

$-e^{-z} = \ln u$

$e^{-z} = -\ln u$

$\boxed{z = -\ln(-\ln u)}$

So, $z_0 = -\ln(-\ln(u_0)) = -\ln(-\ln(0.1))$

$z_1 = -\ln(-\ln(0.3))$

$z_2 = -\ln(-\ln(0.9))$

Q1) c) $z = \dfrac{x-\mu}{\sigma}$

$$x = \mu + z\sigma$$

$$x_0 = \mu + z_0\sigma = 10 + 2*(-\ln(-\ln(0.1)))$$

$$x_1 = 10 + 2*(-\ln(-\ln(0.3)))$$

$$x_2 = 10 + 2*(-\ln(-\ln(0.9)))$$

Q2)

$$f(x) = \begin{cases} 0 , & x < 2a \\ \dfrac{x-2a}{(b-a)^2} , & 2a < x < a+b \\ \dfrac{2b-x}{(b-a)^2} , & a+b < x < 2b \\ 0 , & x \geq 2b \end{cases}$$

In order to apply inverse transform method, we should find cdf of this distribution.

for $2a < x < a+b$ $\quad F(x) = \displaystyle\int_{2a}^{x} \frac{t-2a}{(b-a)^2}dt = \frac{1}{(b-a)^2}\left(\frac{t^2}{2} - 2at\right)\Bigg|_{2a}^{x}$

$$= \frac{1}{(b-a)^2}\left[\left(\frac{x^2}{2} - 2ax\right) - \underbrace{\left(\frac{4a^2}{2} - 4a^2\right)}_{-2a^2}\right]$$

$$= \frac{1}{(b-a)^2}\left[\frac{x^2 - 4ax + 4a^2}{2}\right]$$

$$= \frac{(x-2a)^2}{2(b-a)^2}$$

for $a+b < x < 2b$ $\quad F(x) = \int_{2a}^{a+b} \frac{x-2a}{(b-a)^2}\,dx + \int_{a+b}^{x} \frac{2b-t}{(b-a)^2}\,dt$

$$F(x) = \frac{1}{(b-a)^2}\left(\int_{2a}^{a+b}(x-2a)\,dx + \int_{a+b}^{x}(2b-t)\,dt\right)$$

$$= \frac{1}{(b-a)^2}\left[\left(\frac{x^2}{2} - 2ax\right)\Big|_{2a}^{a+b} + \left(2bt - \frac{t^2}{2}\right)\Big|_{a+b}^{x}\right]$$

$$= \frac{1}{(b-a)^2}\left[\left(\frac{(a+b)^2}{2} - 2a(a+b) - \left(\frac{(2a)^2}{2} - 2a(2a)\right)\right) + \left(2bx - \frac{x^2}{2} - \left(2b(a+b) - \frac{(a+b)^2}{2}\right)\right)\right]$$

$$= \frac{1}{(b-a)^2}\left[\frac{(a+b)^2}{2} - \cancel{2a^2} - 2ab - \cancel{2a^2} + \cancel{4a^2} + 2bx - \frac{x^2}{2} - 2ab - 2b^2 + \frac{(a+b)^2}{2}\right]$$

$$= \frac{1}{(b-a)^2}\left[(a+b)^2 - 4ab - 2b^2 + 2bx - \frac{x^2}{2}\right]$$

$$= \frac{1}{2(b-a)^2}\,2\left[a^2 + 2ab + b^2 - 4ab - 2b^2 + 2bx - \frac{x^2}{2}\right]$$

$$= \frac{1}{2(b-a)^2}\left[2a^2 - 4ab - 2b^2 + 4bx - x^2\right]$$

add $(-2b^2 + 2b^2)$

$$= \frac{1}{2(b-a)^2}\left[2(b-a)^2 - (2b-x)^2\right]$$

$$= 1 - \frac{(2b-x)^2}{2(b-a)^2}$$

So, the cdf is

$$F(x) = \begin{cases} 0 & , x < 2a \\ \dfrac{(x-2a)^2}{2(b-a)^2} & , 2a \leq x < a+b \\ 1 - \dfrac{(2b-x)^2}{2(b-a)^2} & , a+b \leq x < 2b \\ 1 & , x \geq 2b \end{cases}$$

If $x$ is in the interval $(2a, a+b)$ then its cdf is $F(x) = \dfrac{(x-2a)^2}{2(b-a)^2} = u$

$$x = 2a + (b-a)\sqrt{2u} \rightarrow 2a \leq x < a+b$$

$$2a \leq 2a + (b-a)\sqrt{2u} < a+b$$

$$0 \leq (b-a)\sqrt{2u} < (b-a)$$

$$0 \leq \sqrt{2u} < 1$$

$$0 \leq 2u < 1$$

$$0 \leq u < \frac{1}{2}$$

So if $u < \frac{1}{2}$, $x = 2a + (b-a)\sqrt{2u}$

If $x$ is in the interval $(a+b, 2b)$ then its cdf is $F(x) = 1 - \dfrac{(2b-x)^2}{2(b-a)^2} = u$

$$(2b-x)^2 = 2(b-a)^2(1-u)$$

$$x = 2b - (b-a)\sqrt{2(1-u)}$$

$$a+b \leq x < 2b$$

$$a+b \leq 2b - (b-a)\sqrt{2(1-u)} < 2b$$

$$b-a \leq -(b-a)\sqrt{2(1-u)} < 0$$

$$0 \leq (b-a)\sqrt{2(1-u)} < (b-a)$$

$$0 \leq \sqrt{2(1-u)} < 1$$

$$0 \leq 2(1-u) < 1$$

$$0 \leq 1-u < \frac{1}{2}$$

$$-1 \leq -u < -\frac{1}{2}$$

$$\frac{1}{2} \leq u < 1$$

So, if $u \geqslant \frac{1}{2}$ , $x = 2b - (b-a)\sqrt{2(1-u)}$