Question 1 (5 + 5 = 10 pts) Given the line \( \ell \),

\[
\ell : x = 1 - 2t, \quad y = 2 + 5t, \quad z = -3t, \quad t \in \mathbb{R}
\]

and the two planes \( P_1 \) and \( P_2 \),

\[
P_1 : 2x + y - z = 8 \quad \text{and} \quad P_2 : x + y + z = 1.
\]

(a) Determine whether \( \ell \) is parallel or not to the planes \( P_1 \) and \( P_2 \), respectively. Give reasons for your answers.

- Direction vector of \( \ell \) : \( \vec{v} = (-2, 5, -3) \)
- Normal vector of \( P_1 \) : \( \vec{n}_1 = (2, 1, -1) \)
- Normal vector of \( P_2 \) : \( \vec{n}_2 = (1, 1, 1) \)

Recall that \( \vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0 \)

- Since \( \vec{v} \cdot \vec{n}_1 = 4 \neq 0 \), \( \vec{v} \) is not perpendicular to \( \vec{n}_1 \), that is \( \ell \) is not parallel to \( P_1 \)
- Since \( \vec{v} \cdot \vec{n}_2 = 0 \), \( \vec{v} \) is perpendicular to \( \vec{n}_2 \), that is \( \ell \) is parallel to \( P_2 \)

(b) Find the points at which \( \ell \) meets the planes \( P_1 \) and \( P_2 \), respectively. Explain your answers.

- Since \( \ell \) is parallel to \( P_2 \), they do not intersect

- To find the intersection point of \( \ell \) and \( P_1 \), substitute \( x = 1 - 2t, \quad y = 2 + 5t, \quad z = -3t \) in the equation of \( P_1 \):

\[
2(1 - 2t) + (2 + 5t) - (-3t) = 8
\]

\[
4 + 4t = 8 \quad \Rightarrow \quad t = 1
\]

\( \Rightarrow \quad (x, y, z) = (-1, 7, -3) \) is the intersection point of \( \ell \) and \( P_1 \).
Question 2 (10 pts) Find the plane, which is parallel to the tangent plane to the surface \(3x - x^2y^2 + \sin(xz) = 2\) at \((1, 1, 0)\), and which passes through the point \((2, 2, 2)\).

Given a surface \(z = f(x, y)\), normal vector to the tangent plane is \(\mathbf{n} = \langle f_x, f_y, -1 \rangle\). To find \(f_x = \frac{2}{2x}\) and \(f_y\), use implicit differentiation: 
\[
3 - 2xy^2 + x \cos(xz) + x \cos(xz) \frac{\partial z}{\partial x} = 0 \\
\text{at } x = 1, y = 1, z = 0 \quad \text{we have} \quad 1 + 2z = 0 \quad \Rightarrow \quad z = -1
\]

Similarly: 
\[-2x^2y + x \cos(xz) \cdot 2y = 0 \quad \Rightarrow \quad -2x^2y = 0 \quad \Rightarrow \quad z = 2
\]
So \(\mathbf{n} = \langle -1, 2, -1 \rangle\), \(P_0 = (2, 2, 2)\)

Tangent Plane: 
\[
-\mathbf{n} \cdot \langle x-2, y-2, z-2 \rangle = 0 \quad \Rightarrow \quad \boxed{\mathbf{n} \cdot \langle x-2, y-2, z-2 \rangle = 0}
\]

Only given a surface \(f(x, y, z) = 0\), \(\nabla f(1, 1, 0)\) is perpendicular to the tangent plane at \((1, 1, 0)\).

Since \(f(x, y, z) = 3x - x^2y^2 + \sin(xz) - 2 = 0\),
\[
\nabla f(x, y, z) = \langle 3, -2xy^2 + x \cos(xz) + x \cos(xz), x \cos(xz) \rangle \\
\nabla f(1, 1, 0) = \langle 3, 0, 0 \rangle
\]

Take \(\mathbf{n} = \nabla f(1, 1, 0) \quad \Rightarrow \quad \text{Eqn. of tangent plane} \quad x - 2y + 2z = 0\)

Question 3 (10 pts) Find the rate of change of \(F(x, y, z) = 3e^z \cos(yz)\) at the origin in the direction of the vector \(v = (2, 1, -2)\). In what directions is the rate of change of \(F(x, y, z)\) at \((0, 0, 0)\) equal to zero?

Let \(\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{9}} \mathbf{v} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle\) be the unit vector in direction of \(\mathbf{v}\)

\[
\nabla f(x, y, z) = \langle 3e^z \cos(yz), -3ze^z \sin(yz), -3yze^z \sin(yz) \rangle
\]

\[
\nabla f(0, 0, 0) = \langle 3, 0, 0 \rangle
\]

Since \(F\) is a polynomial, \(F_x, F_y, F_z\) and \(D_u F\) all exist at \((0, 0, 0)\) and hence we can use the formula:

\[
D_u F(0, 0, 0) = \nabla f(0, 0, 0) \cdot \mathbf{u} = 2
\]

Again from the above formula, we see that \(D_u F(0, 0, 0) = 0\)

if \(\nabla f(0, 0, 0) \perp \mathbf{w}\)

Let \(\mathbf{w} = \langle w_1, w_2, w_3 \rangle\), \(\|\mathbf{w}\| = 1\),

then \(\nabla f(0, 0, 0) \cdot \mathbf{w} = 3w_1 = 0 \quad \Rightarrow \quad w_1 = 0 \)

In direction of the vectors of the form \(\mathbf{w} = \langle 0, w_2, w_3 \rangle\) with \(w_2^2 + w_3^2 = 1\),
rate of change of \(F\) at the origin is zero.
Question 4 \(3 + 6 + 6 = 15\) pts) Consider the lines \((1 + 2t, 2 + 3t, 3 + 4t)\) and \((2 + s, 4 + 2s, -1 - 4s)\), where \(t\) and \(s\) are parameters.

(a) Find the cosine of the angle between two lines.

**Direction vector of the first line:** \(\vec{v}_1 = \langle 2, 3, 4 \rangle\)

**Direction vector of the second line:** \(\vec{v}_2 = \langle 1, 2, -4 \rangle\)

\[
\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos \theta \Rightarrow 
\cos \theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{8}{\sqrt{29}}(21)
\]

(b) Find the point of intersection of the lines.

\[
\begin{align*}
1 + 2t &= 2 + s \\
2 + 3t &= 4 + 2s
\end{align*}
\Rightarrow \begin{align*}
s &= 2t - 1 \\
4 + 2(2t-1) &= 2 + 4t
\end{align*}
\Rightarrow t = 0
\Rightarrow s = -1
\]

At \(t = 0\) and \(s = -1\), \(x\) and \(y\) coordinates coincide.

Note also that \(t = 0\) as \(3 + 4t = 3\) hence \(z\) coordinates \(s = -1\) \(\Rightarrow -1 - 4s = 3\) also coincide.

Thus \(x = 1\), \(y = 2\), \(z = 3\), that is \(P(1, 2, 3)\) is the intersection point of the lines.

(c) Find the plane determined by these lines.

**Normal vector** \(\vec{n}\) of the plane is parallel to \(\vec{v}_1 \times \vec{v}_2\)

\[
\vec{n} \parallel \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
2 & 3 & 4 \\
1 & 2 & -4
\end{vmatrix} = (-12 - 8)\vec{i} - (-8 - 4)\vec{j} + (4 - 3)\vec{k} = \langle -20, 12, 1 \rangle
\]

We can take \(\vec{n} = \langle -20, 12, 1 \rangle\)

\(P_0(1, 2, 3)\)

Plane: \(-20(x-1) + 12(y-2) + 4z = 0 \Rightarrow -20x + 12y + 2z = 4\)
Question 5 (5 pts) Evaluate the limit or explain why it does not exist

\[
\lim_{(x,y) \to (0,0)} \frac{x^6 - 2x^3y^3 + y^6}{x^6 + y^6} = \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x^6 - 2m^3x^6 + m^6x^6}{x^6 + m^6x^6} = \frac{1-2m^3+m^6}{1+m^6}
\]

As \((x,y)\) gets closer to \((0,0)\) along the line \(y = mx\), we have since result depends on \(m\) \((m = 1 \Rightarrow \text{it is 0}, m = -1 \Rightarrow \text{it is 2})\), limit does not exist.

Question 6 (10 pts) Let \(z = f(x, y) = e^{xy}\) be a function of two variables, where \(z = r^2 - 4s\) and \(y = \sin r + s^3\). Use chain rule differentiation to find \(\frac{\partial z}{\partial r}\) and \(\frac{\partial z}{\partial s}\) in terms of \(r\) and \(s\).

\[
\frac{\partial z}{\partial r} = y e^{xy} 2r + xe^{xy} \cos r
\]

\[
\frac{\partial z}{\partial s} = (r^2 - 4s)(\sin r + s^3) \left[ 2r (\sin r + s^3) + \cos r (r^2 - 4s) \right]
\]

\[
\frac{\partial z}{\partial s} = \frac{2f}{\partial s} = \frac{2x}{\partial s} + \frac{2f}{\partial y} \frac{2y}{\partial s} = y e^{xy} (-4) + xe^{xy} (3s^2)
\]

\[
= (r^2 - 4s)(\sin r + s^3) \left[ -4 (\sin r + s^3) + 3s^2 (r^2 - 4s) \right]
\]