MATH 463

October 29, 2014

1. HOMEWORK SOLUTIONS

1.1. *Assume that a set G with an operation satisfying the associative law satisfies the following two conditions (a) and (b):*

(a) There exists an element e of G such that $ge = g$ for all $g \in G$.

(b) For any element a of G, there exists an element a' such that $aa' = e$.
Then, show that G is a group with respect to the given operation.

Solution We need to show that there exists a left identity and each element has a left inverse. Apply (b) to the element a'. So there exists $a'' \in G$ with $a'a'' = e$. By the associative law;
$ea'' = (aa')a'' = a(a'a'') = ae = a$ by part (a). So we have $ea'' = a$

On the other hand; $ea = (ea)e = (ea)(a'a'') = e(aa')a'' = (ee)a'' = ea'' = a$ by the above paragraph.
Therefore for any element $a \in G$ we have $ea = a = ae$ for all $a \in G$. So, e is the identity element of G.
Since we have $ea'' = a$ and e is the identity element, we get $a'' = a$. So we have $aa' = e$ and $a'a'' = a'a = e = aa'$. So a' is the inverse of a.
Therefore, G is a group with the given conditions.

1.2. *For a given subset X of a group G, let $\mathscr{H}$ be the set of subgroups H satisfying $H \cap X = \emptyset$ (the empty set). The set $\mathscr{H}$ becomes a partially ordered set by defining $H \leq K$ if and only if H and K are members of $\mathscr{H}$ and H is a subgroup of K. Show that, if $\mathscr{H}$ is not empty, $\mathscr{H}$ is inductively ordered, so $\mathscr{H}$ has at least one maximal element by Zorn's lemma.*
Pick a subgroup H_0 satisfying $H_0 \cap X = \emptyset$, and let $\mathscr{H}_0$ denote the subset of $\mathscr{H}$ consisting of the members which contain H_0. Show that $\mathscr{H}_0$ is also inductively ordered, and has a maximal element.

Solution Assume $\mathscr{H}$ is non-empty. It is clear that $\mathscr{H}$ is a partially ordered set as being a subgroup is a partially ordered set on the set of all subgroups of G. This is the restriction of this relation to $\mathscr{H}$. Since $\mathscr{H} \neq \emptyset$, there exists a subgroup $H_0 \in \mathscr{H}$ such that $H_0 \cap X = \emptyset$. Let

$$\mathscr{H}_0 = \{H \in \mathscr{H} |\ H_0 \leq H\ \}$$

Let $H_i, i \in I$ be a chain of subgroups in $\mathscr{H}_0$. Then $T = \bigcup_{i \in I} H_i$ is a subgroup of G and $T \in \mathscr{H}_0$ as $T \cap X = \emptyset$. Hence every ascending chain of members in $\mathscr{H}_0$ has an upper bound in $\mathscr{H}_0$. Then by Zorn's lemma there exists a maximal element in $\mathscr{H}_0$. i.e. There exists a subgroup M of G M is a maximal element in $\mathscr{H}_0$. Therefore every subgroup containing M will have a non-empty intersection.

1.3. *Find the number of left cosets of K which are contained in the double coset HxK, also show that G is the disjoint union of its (H, K)-double cosets.*

Solution

1.4. *Let H be a proper subgroup of a finite group G. Show that there exists an element of G which is not conjugate to any element of H.*

Solution Assume for any $x \in G$, there exists $g \in G$ such that $x \in H^g$. Then $G = \bigcup H^g$. Let $|G| = n$ and $|H| = k$.
The number of distinct conjugates of H is $[G : N_G(H)]$.

Then we have $|G| = [G : N_G(H)]|N_G(H)| \geq [G : N_G(H)]|H|$ as $N_G(H) \geq H$. Let $|G : N_G(H)| = m$. Then H has m distinct conjugates in G. Say $H = H^1, H^{g_2}, \ldots, H^{g_m}$ As each H^{g_i} contain $|H| - 1$ non-identity element we have at most $|H^{g_i}| - 1$ non-identity element in H^{g_i}. If $G = \bigcup_{i=1}^m H^{g_i}$. Then $|G| = \sum_{i=1}^m |H^{g_i}| \leq \sum_{i=1}^m (|(H^{g_i} - id)| \leq (k-1)m + 1$ as $H \leq N_G(H)$ we have $mk - m + 1 \geq |G| = m(|N_G(H)| \geq mk$.

So we have $-m+1 \geq 0$ and $m \leq 1$. But $m = 1$ implies that $H \lhd G$ and in this case $H^g = H$ for all $g \in G$. This implies that $H = G$. This contradicts to the assumption that H is a proper subgroup of G. So G cannot be a union of conjugates of a proper subgroup H.

1.5. ***(a)*** *Prove that any subgroup of index 2 is normal.*
(b) *Let G be a finite group, and let p be the smallest prime divisor of the order $|G|$. Show that any subgroup of index p is normal.*

Solution (a) Let $H \leq G$ with $[G : H] = 2$.
Then H has two distinct right cosets, and also two distinct left cosets in G. For any $h \in H$, we have $hH = Hh = H$ and for any $a \in G$ with $a \notin H$, we have $aH \neq H$ and $Ha \neq H$. Since there are exactly two cosets of H in G, we have $Ha = aH = G \setminus H$ for all $a \in G$.
Therefore $H \trianglelefteq G$.

(b) Let H be a subgroup of G of index p. Then we need to show that H is a normal subgroup of G. Indeed G acts from right on the set of right cosets of H in G. Then there exists a homomorphism from G into $Sym(p)$. Then $G/Ker(\phi)$ is isomorphic to a subgroup of $Sym(p)$. Recall that $Ker(\phi) = \bigcap_{x \in G} H^x$. So $Ker(\phi) \leq H$. If H is not normal in G then $Ker(\phi)$ will be a proper subgroup of H and hence $1 \neq H/Ker(\phi) < G/Ker(\phi)$. i.e a prime divisor of $|H/Ker(\phi)|$ divides $|G|/|Ker(\phi)|$ which divides $\frac{p!}{|Ker(\phi)|}$. Hence it divides $|G|$ which is impossible as any prime dividing $p!$ is less than p and p is the smallest prime dividing $|G|$.

1.6. *For any proper subgroup H of a group G, $HH^x \neq G$ for any $x \in G$.*

Solution Assume that $HH^x = G$ for some $x \in G$. Since H is a proper subgroup, clearly $x \neq 1$. Then $x = h_1 h_2^x$ for some $h_1, h_2 \in H$. Then $x = h_1 x^{-1} h_2 x$. It follows that $1 = h_1 x^{-1} h_2$ and so $h_1^{-1} h_2^{-1} = x^{-1}$. Since H is a subgroup and $h_1, h_2 \in H$ we have $h_1^{-1} h_2^{-1} \in H$ i.e. $x \in H$. But then, $G = HH^x = H$. This contradicts to H is proper. Hence $HH^x \neq G$.

2. HOMEWORK SOLUTIONS

Definition 2.1. *An endomorphism σ of a group G is said to be normal if σ commutes with all inner automorphisms of G.*

2.1. *Let σ be a normal endomorphism of a group G. Set $\sigma(G) = H$ and $\sigma(g) = z(g)^{-1}g$ for any $g \in G$.*

(a) *Show that z is a homomorphism from G into $C_G(H)$.*

(b) *Show that H is a normal subgroup of G such that $G = HC_G(H)$, and $H \cap C_G(H) = Z(H) \subset Z(G)$.*

(c) *Show that both H and $C_G(H)$ are invariant by σ. Prove that the restriction ρ of σ on $C_G(H)$ is a homomorphism from $C_G(H)$ into $Z(H)$, and that for any element x of $Z(H)$, we have $x = \zeta(x)\rho(x)$ where ζ is the restriction of z on H.*

(d) *Conversely, suppose that $G = HC_G(H)$, and that a homomorphism $\zeta : H \to Z(H)$ and a homomorphism $\rho : C_G(H) \to Z(H)$ are given so that, for any element x of $Z(H)$, the formula $x = \zeta(x)\rho(x)$ is satisfied. Prove that, for an element g of G, the formula*

$$\sigma(g) = \rho(c)\zeta(h)^{-1}h,$$

where $g = hc, h \in H$ and $c \in C_G(H)$, defines a normal endomorphism from G into H.

Solution

(a) Let σ be a normal endomorphism of a group G. Then σ is an endomorphism of G, commuting with all the inner automorphisms of G. Let $\sigma(G) = H$ and $\sigma(g) = z(g)^{-1}g$. We may view this as $z(g) = g\sigma(g)^{-1}$.

First observe that $z(g) = g\sigma(g)^{-1} \in C_G(H)$. Indeed;

$i_g\sigma = \sigma i_g$ implies for any $x \in G$ $((x)i_g)\sigma = ((x)\sigma)i_g$. Then $(g^{-1}xg)\sigma = g^{-1}((x)\sigma)g$. It follows that

$((g^{-1})\sigma)((x)\sigma)((g)\sigma) = g^{-1}((x)\sigma)g$. Multiply from left by g and from right by (g^{-1}) we have $[g((g^{-1})\sigma)]((x)\sigma)(g)\sigma)g^{-1} = (x)\sigma$ for any $x \in G$. So for any $(x)\sigma \in H$ we have $z(g) = g(g^{-1})\sigma \in C_G((G)\sigma) = C_G(H)$

Now for any g and h in G;

$(gh)z = gh((gh)\sigma)^{-1} = gh((g)\sigma(h)\sigma)^{-1} = gh((h)\sigma)^{-1}((g)\sigma)^{-1}$

By first paragraph $h(h^{-1})\sigma \in C_G((G)\sigma)$ so $h(h^{-1})\sigma$ commutes with $(g^{-1})\sigma$ and we obtain

$(gh)z = g((g^{-1})\sigma)h((h^{-1})\sigma) = (g)z(h)z$ Hence z is a homomorphism from G into $C_G(H)$.

(b) $H = (G)\sigma$. For any $g \in G$ and $(x)\sigma \in H$

$g^{-1}(x)\sigma g = g^{-1}(x)\sigma g((g)\sigma)^{-1}(g)\sigma$ as $g((g)\sigma)^{-1} \in C_G(H)$ we have

$= g^{-1}g((g)\sigma)^{-1}(x)\sigma(g)\sigma$

$= ((g)\sigma)^{-1}(x)\sigma(g)\sigma = (g^{-1}xg)\sigma \in H$

So H is a normal subgroup of G.

Now for any $g \in G$

$g = (g)\sigma g((g)\sigma)^{-1}$ as $g((g)\sigma)^{-1} \in C_G(H)$ and $(g)\sigma \in H$ we have

$G = HC_G(H)$ and $H \cap C_G(H) = Z(H)$.

If $x \in H \cap C_G(H)$, then for any $g \in G$

$gx = (g)\sigma g((g^{-1})\sigma)x$

$= (g)\sigma x g((g^{-1})\sigma)$ as $x \in H$ and $g((g^{-1})\sigma) \in C_G(H)$

$= x(g)\sigma g((g^{-1})\sigma)$ as $x \in C_G(H)$ and $(g)\sigma \in H$.

$= xg$. So $x \in Z(G)$ and $Z(H) = H \cap C_G(H) \leq Z(G)$.

(c) **(i)** H is invariant as $(H)\sigma = ((G)\sigma)\sigma \subseteq (G)\sigma = H$

Let $x \in C_G(H)$. Then for any $h \in H, xh = hx$.

i.e. $x(g)\sigma = (g)\sigma x$ for any $g \in G$. $x(g)\sigma x^{-1} = (g)\sigma$ for all $g \in G$.

Consider $(x)\sigma(g)\sigma = (g)\sigma(x)\sigma$?

$(x)\sigma x^{-1}x(g)\sigma = (x)\sigma x^{-1}(g)\sigma x$

$= (g)\sigma(x)\sigma x^{-1}x$ as $(x)\sigma x^{-1} = (x(x^{-1})\sigma)^{-1} \in C_G(H)$ and $(g)\sigma \in H$

$= (g)\sigma(x)\sigma$

Hence $(x)\sigma \in C_G(H)$.

(ii) The restriction ρ:

Let $x, y \in C_G(H)$. Then $(x)\rho = (x)\sigma = ((x)z)^{-1}x$. $((x)z)^{-1}x \in Z(H)$ as for any $(g)\sigma \in H$, we have $((x)z)^{-1}x(g)\sigma = ((x)z)^{-1}(g)\sigma x$ as $x \in C_G(H)$ and $(g)\sigma \in H$.

as $(x)z \in C_G(H)$ we have $((x)z)^{-1}x(g)\sigma = (g)\sigma((x)z)^{-1}x$.

It follows that $((x)z)^{-1}x \in Z(H)$ and $(x)\rho \in Z(H)$.

Moreover $(xy)\rho = (xy)\sigma = (x)\sigma(y)\sigma = (x)\rho(y)\rho$

(iii) Let $x \in Z(H)$. Then $x = x((x)\sigma)^{-1}(x)\sigma$.

Now $x((x)\sigma)^{-1} = (x)z = (x)\zeta$ where ζ is the restriction of z on H. And $(x)\sigma = (x)\rho$ where ρ is the restriction of σ on $C_G(H)$.

55.1 Let G be a group with $Z(G)$=1. Show that the centralizer in $Aut(G)$ of $Inn(G)$ is $\{1\}$ and in particular, $Z(Aut(G))$=$\{1\}$.

Solution: Let $\phi \in C_{Aut(G)}(Inn(G))$. Then

$\phi^{-1} i_g \phi = i_g$ for any $i_g \in Inn(G)$. For any element $x \in G$, $\phi^{-1} i_g \phi(x) = i_g(x)$ and so $\phi^{-1} i_g(\phi(x)) = g^{-1}xg$. It follows that $\phi^{-1}(g^{-1}\phi(x)g) = g^{-1}xg$ iff $\phi^{-1}(g^{-1})x\phi^{-1}(g) = g^{-1}xg$. Then we have

$g\phi^{-1}(g^{-1})x\phi^{-1}(g)g^{-1} = x$. Hence

$(g^{-1})^{-1}(\phi^{-1}(g))^{-1}x\phi^{-1}(g)g^{-1} = x$ for all $x \in G$.

Hence, $\phi^{-1}(g)g^{-1} \in Z(G) = 1$. It follows that $\phi^{-1}(g) = g$ for all $g \in G$. Then the automorphism fixes all the elements of G. i.e. ϕ is the identity automorphism of G.

As $Z(Aut(G)) = C_{Aut(G)}(Aut(G)) \leq C_{Aut(G)}(Inn(G))$=1, we have $Z(Aut(G))$=1.

56.3 Let G be a nonabelian simple group. Show that any automorphism of $Aut(G)$ is inner.

Solution: As G is nonabelian simple group, $Z(G)$=1. Then by 55.1, $Z(Aut(G))$=1. Then by 55.2, any automorphism of $A = Aut(G)$ is an inner automorphism.

Question: If two subgroups H and K of a group G satisfy the conditions $H \cap K = \{1\}$, $H \leq N_G(K)$ and $K \leq N_G(H)$, then every element of H commutes with every element of K.

Solution: Consider the element $h^{-1}k^{-1}hk$. Since $K \leq N_G(H)$, $k^{-1}hk \in H$. So $h^{-1}k^{-1}hk \in H$. Similarly, $H \leq N_G(K)$ implies $k^{-1}hk \in K$. So $h^{-1}k^{-1}hk \in K$. Hence, $h^{-1}k^{-1}hk \in H \cap K = \{1\}$. It follows that $h^{-1}k^{-1}hk = 1$ and $hk = kh$ for any $h \in H$ and $k \in K$.

45.1 Let G be a group with a composition series and let N be a normal subgroup of G. Show that there is a composition series of G

having N as a term.

Solution: Let G be a group with a composition series $G = G_0 \triangleright G_1 \triangleright .. \triangleright G_n = \{1\}$.

Take the intersection of each subgroup in the series with the normal subgroup N. We have $G_0 \cap N = N \triangleright G_1 \cap N \triangleright G_2 \cap N \triangleright .. \triangleright G_n \cap N = \{1\}$.

Now, we need to show $G_{i+1} \cap N \trianglelefteq G_i \cap N$. Indeed, let $x \in G_{i+1} \cap N$ and $g \in G_i \cap N$. Then $g^{-1}xg \in N$ as $x \in N$ an N is a normal subgroup of G.

Moreover, $x \in G_{i+1}$ and $g \in G_i$ and G_{i+1} is normal in G_i implies $g^{-1}xg \in G_{i+1}$. Hence, $x \in G_{i+1} \cap N$ and so $G_{i+1} \cap N \trianglelefteq G_i \cap N$.

$(G_i \cap N)/(G_{i+1} \cap N) \simeq (G_i \cap N)G_{i+1}/G_{i+1} \trianglelefteq G_i/G_{i+1}$.

But G_i/G_{i+1} is a composition factor of the group G. So $(G_i \cap N)/(G_{i+1} \cap N)$ is either equal to G_i/G_{i+1} or $\{1\}$.

So it is simple or $(G_i \cap N)G_{i+1}/G_{i+1}$ is the trivial group.

So N has a series where each factor is either simple and the simple factor is isomorphic to a simple factor of G or it is trivial group. By deleting the trivial terms from the series, we obtain a composition series of N.

Now we may look at the series $G \triangleright G_1N \triangleleft G_2N \ldots N$ this series also give a series from G to N with factors are either trivial or simple apply the same procedure above and obtain a series of G.