	Calculus and Analytical Geometry II. Midterm	
Code : *Math 119* Acad.Year: *2009-2010* Semester : *Fall* Date : *16.12.2009* Time : *17:40* Duration : *120 minutes*	Last Name: Name : KEY Department: Signature: 6 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	Student No.: Section:

1	2	3	4	5	6		

Please show your work in all questions.

1. (5+5+5+5=20 points) Evaluate the following integrals

(a) $\int (x - \frac{1}{x})(\sqrt{x} + \frac{1}{\sqrt{x}})dx = \int (x^{3/2} + x^{1/2} - x^{-1/2} - x^{-3/2})\,dx =$

$= \frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} - 2x^{1/2} + 2x^{-1/2} + C$

(b) $\int \sin^2 t \cos t dt = \left| \begin{matrix} u = \sin(t) \\ du = \cos(t)dt \end{matrix} \right| = \int u^2 du = \frac{u^3}{3} + C =$

$= \frac{\sin^3(t)}{3} + C$

(c) $\int \frac{4x^3}{\sqrt{x^2+1}} dx = \left| \begin{matrix} u = x^2+1 \\ du = 2x\,dx \end{matrix} \right| = 2 \int \frac{u-1}{\sqrt{u}} du = 2 \int (\sqrt{u} - u^{-1/2}) du$

$= 2\left(\frac{2}{3} u^{3/2} - 2u^{1/2}\right) + C = \frac{4}{3}(x^2+1)^{3/2} - 4(x^2+1)^{1/2} + C$

(d) $\int_{-1}^{1} 1 + \frac{x^4 \tan x}{1+x^2} dx. = \int_{-1}^{1} dx + \int_{-1}^{1} \frac{x^4 \tan(x)}{1+x^2} dx = 2 + \int_{-1}^{1} \frac{x^4 \tan(x)}{1+x^2} dx$

The function $f(x) = \frac{x^4 \tan(x)}{1+x^2}$ is odd and we deal with the symmetric interval $[-1,1]$. Using Symmetry Property for the definite integral, we conclude $\int_{-1}^{1} f(x)\,dx = 0.$

2. (1+1+1+4+3+2+4+4=20 points) Follow the outline below to sketch the graph of the function $f(x) = \dfrac{x^3}{9-x^2}$.

(a) Find the domain of $f(x)$.

$$\mathrm{dom}(f) = \mathbb{R} \setminus \{\pm 3\}.$$

(b) Find the intercepts.

$x = 0 \iff y = 0$. So, we have only $(0,0)$.

(c) Find the symmetries of $f(x)$, if any.

$f(-x) = \dfrac{(-x)^3}{9-(-x)^2} = -f(x)$, that is, $f(x)$ is odd.

(d) Determine all asymptotes.

$\dfrac{x^3}{9-x^2} = -x + \dfrac{9x}{9-x^2} \Rightarrow y = -x$ is a slant asymptote.

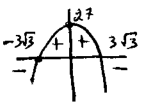
$x=-3$ and $x=3$ are vertical asymptotes, for

$\lim\limits_{x\to -3^-} f(x) = +\infty$, $\lim\limits_{x\to -3^+} f(x) = -\infty$, $\lim\limits_{x\to 3^-} f(x) = +\infty$, $\lim\limits_{x\to 3^+} f(x) = -\infty$.

(e) Find the intervals of increase and decrease, and the critical points.

$$f'(x) = \frac{x^2(27-x^2)}{(9-x^2)^2}$$

$$\mathrm{C.P.}(f) = \{0, \pm 3\sqrt{3}\}$$



x	$x<-3\sqrt{3}$	$-3\sqrt{3}<x<-3$	$-3<x<0$	$0<x<3$	$3<x<3\sqrt{3}$	$3\sqrt{3}<x$
Sign $f'(x)$	−	+	+	+	+	−
Sign $f''(x)$	+	+	−	+	−	−
Behavior $f(x)$	decr. con. up	incr. con. up	incr. con. down	incr. con. up	incr. con. down	decr. con. down

(f) Find all local maxima and minima.

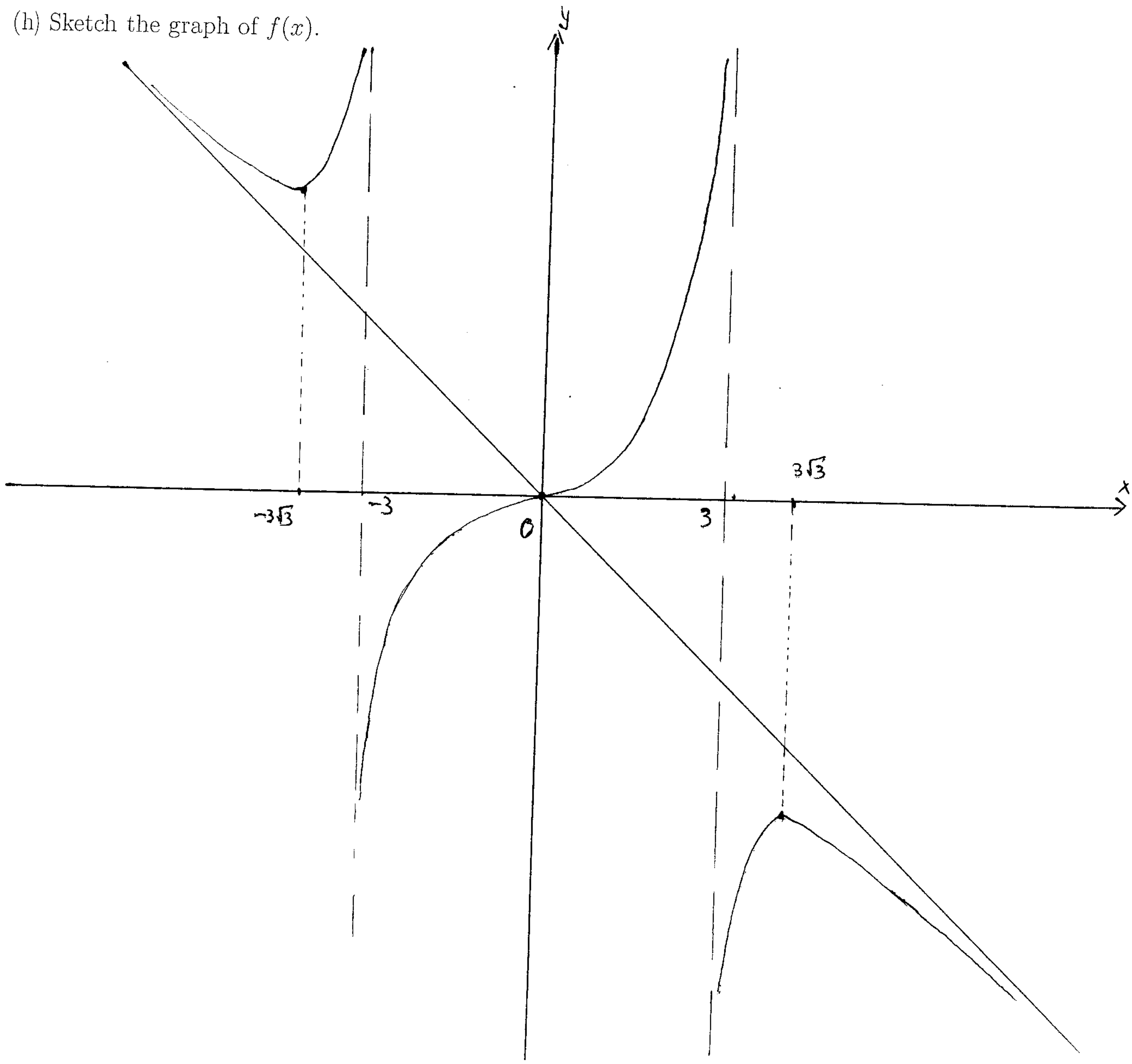
Using F. D. T., we conclude that $x=-3\sqrt{3}$ is a local max., $x = 3\sqrt{3}$ is a local min. points.

(g) Determine the concavity of $f(x)$, and its inflection points.

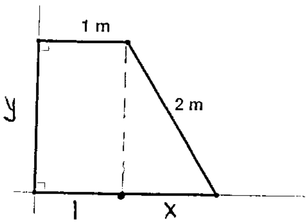
$$f''(x) = \frac{18x(27+x^2)}{(9-x^2)^3}$$

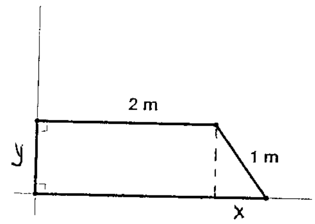
$C.P.(f') = \{0\}$ The point $x=0$ is an inflection point (see to the table)

(h) Sketch the graph of $f(x)$.



3. (17 points) We want to form a right angled trapezoid with one corner at the origin, two of the sides along the positive x and y axes, and the other two sides having lengths of $1m$ and $2m$ (see the figure below). Find the maximum possible area of such a trapezoid.





If A is the area then

$$A = \frac{1}{2}xy + y$$

Since $x^2+y^2=4$, it follows that $y=\sqrt{4-x^2}$, $0\le x\le 2$. Then

$$A = A(x) = \left(\frac{1}{2}x+1\right)\sqrt{4-x^2}, \quad 0\le x\le 2$$

Take the derivative

$$A'(x) = -\frac{x^2+x-2}{\sqrt{4-x^2}}.$$

$$C.P.(A) = \{1, -2\}$$

$$A(1) = \frac{3\sqrt{3}}{2}$$

Since $A(0)=2$, $A(2)=0$, we derive that $A(x)$ takes abs. max. at $x=1$.

$$A = \frac{1}{2}xy + 2y$$

Similarly, $x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$, $0\le x\le 1$. Then

$$A = A(x) = \left(\frac{1}{2}x+2\right)\sqrt{1-x^2}, \quad 0\le x\le 1.$$

Similarly,

$$A'(x) = -\frac{x^2+2x-\frac{1}{2}}{\sqrt{1-x^2}}.$$

$$C.P.(A) = \left\{\frac{\pm\sqrt{6}-2}{2}\right\} = \left\{\pm\sqrt{3/2}-1\right\}$$

$$A\left(\sqrt{\tfrac{3}{2}}-1\right) = \left(\frac{1}{2}\sqrt{\tfrac{3}{2}}+\frac{3}{2}\right)\left(\sqrt{\sqrt{6}-3/2}\right)$$

Since $A(0)=2$, $A(1)=0$ and $A\left(\sqrt{\tfrac{3}{2}}-1\right) > 2$, we conclude that $A(x)$ takes abs. max at $x=\sqrt{\tfrac{3}{2}}-1$.

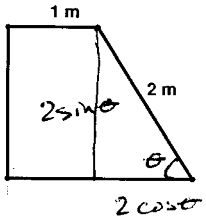
Finally,

$$\frac{3\sqrt{3}}{2} > \left(\frac{1}{2}\sqrt{\tfrac{3}{2}}+\frac{3}{2}\right)\sqrt{\sqrt{6}-\tfrac{3}{2}}$$

maximum possible area

(alternate solution)

3. (17 points) We want to form a right angled trapezoid with one corner at the origin, two of the sides along the positive x and y axes, and the other two sides having lengths of $1m$ and $2m$ (see the figure below). Find the maximum possible area of such a trapezoid.



If A is the area then

$$A = 2\sin\theta + \frac{4\sin\theta\cos\theta}{2}$$

$$A = 2\sin\theta + 2\sin\theta\cos\theta$$

for $0 \le \theta \le \pi/2$.

Critical points:

$$0 = A' = 2\cos\theta + 2\cos^2\theta - 2\sin^2\theta$$

$$0 = 4\cos^2\theta + 2\cos\theta - 2$$

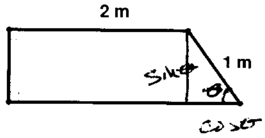
So $\cos\theta = \dfrac{-1 \pm \sqrt{1+8}}{4} = \dfrac{-1 \pm 3}{4}$

$\cos\theta = -1 \Downarrow \theta = \pi$ | $\boxed{\cos\theta = 1/2 \Downarrow \theta = \pi/3}$

$$A(\pi/3) = 2\frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{3\sqrt{3}}{2}$$

$A(0) = 0$ & $A(\pi/2) = 2$

so this is a max.



$$A = 2\sin\theta + \frac{\sin\theta\cos\theta}{2}$$

Critical points:

$$0 = A' = 2\cos\theta + \tfrac{1}{2}\cos^2\theta - \tfrac{1}{2}\sin^2\theta$$

$$0 = \cos^2\theta + 2\cos\theta - 1/2$$

So $\cos\theta = \dfrac{-2 \pm \sqrt{4+2}}{2} = -1 \pm \sqrt{3/2}$

$\cos\theta = -1 - \sqrt{3/2} \Downarrow$ Impossible! | $\cos\theta = \sqrt{3/2} - 1 \Downarrow$

$$\sin\theta = \sqrt{1 - (\sqrt{3/2} - 1)^2} = \sqrt{\sqrt{6} - 3/2}$$

Then

$$A = 2\sqrt{\sqrt{6} - 3/2} + \tfrac{1}{2}\sqrt{\sqrt{6} - 3/2}\left(\sqrt{3/2} - 1\right)$$

$$= \sqrt{\sqrt{6} - 3/2}\left(\tfrac{1}{2}\sqrt{3/2} + 3/2\right)$$

This is clearly bigger than 2 so it is also a max.

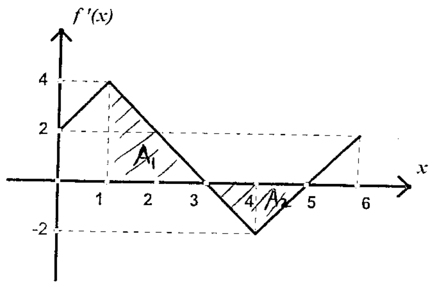
$$\frac{3\sqrt{3}}{2} > \sqrt{\sqrt{6} - 3/2}\left(\tfrac{1}{2}\sqrt{3/2} + 3/2\right)$$

4. (6+(2+2+2)+6=18 points)

(a) Find the derivative of $\int_0^{x^2} \tan(\cos(t))dt$ with respect to x.

Using the Chain Rule, we obtain that

$$\frac{d}{dx}\int_0^{x^2} \tan(\cos(t))dt = \tan(\cos(x^2))\cdot 2x$$



(b) Suppose that $f'(x)$ has the graph shown in the figure to the right. Fill in the blanks in the following three questions accordingly:

(i) If $f(1) = 4$, then $f(5) =$...6.... . Indeed,

$$f(5) = f(1) + \int_1^5 f'(x)dx = 4 + A_1 - A_2 = 4 + 4 - 2 = 6.$$

(ii) $f(x)$ has a local maximum (or maxima) at ...3.... and a local minimum (or minima) at ....5..

The points where $f'(x)$ changes its sign.

(iii) $f(x)$ has an inflection point (or points) at ..1..and 4.

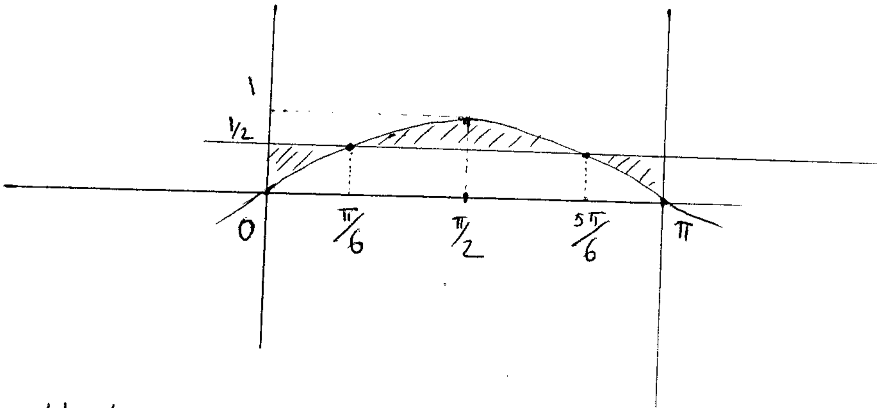
The critical points for $f'(x)$, where we have loc. max and loc. min., respectively.

(c) Express $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sin(i/n)}{n}$ as a definite integral.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sin\left(\frac{i}{n}\right) = \int_0^1 \sin(x)dx$$

thanks to the definition of Riemann sums.

5. (15 points) Sketch and find the area between the curves $y = \sin x$ and $y = \frac{1}{2}$ between $x = 0$ and $x = \pi$. (Note that the region has three pieces.)



Note that $\sin(x) = \frac{1}{2}$ in $[0, \pi]$ iff $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

Using the symmetry, we conclude that

$$\frac{1}{2}A = \int_{\pi/2}^{5\pi/6} \left(\sin(x) - \frac{1}{2}\right) dx + \int_{5\pi/6}^{\pi} \left(\frac{1}{2} - \sin(x)\right) dx =$$

$$= -\cos(x) \Big|_{\pi/2}^{5\pi/6} - \frac{1}{2}x \Big|_{\pi/2}^{5\pi/6} + \frac{1}{2}x \Big|_{5\pi/6}^{\pi} + \cos(x) \Big|_{5\pi/6}^{\pi}$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2} = \sqrt{3} - 1 - \frac{\pi}{12}.$$

Hence $A = 2\sqrt{3} - 2 - \pi/6$.

6. (10 points) Suppose that a twice differentiable function $f(x)$ satisfies the equation $f''(x) = (f(x))^2 + 1$ for all x. Show that $f(x)$ has at most one critical point.

Assume that $f(x)$ has two different critical points, say $\underline{\underline{a}}$ and $\underline{\underline{b}}$ $(a<b)$.
So, $f'(a) = f'(b) = 0$. Using M. V. T.,
we derive that

$$\frac{f'(b) - f'(a)}{b - a} = f''(c)$$

for some point $c \in (a, b)$. Then
$f''(c) = 0$. But $f''(c) = f(c)^2 + 1 \geq 1$,
a contradiction.
Hence $a = b$.