M E T U
Northern Cyprus Campus

	Calculus and Analytical Geometry I. Midterm	
Code : *Math 119* Acad.Year: *2009-2010* Semester : *Fall* Date : *3.11.2008* Time : *17:40* Duration : *120 minutes*	Last Name: Name : KEY Department: Signature: 8 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	Student No Section:

1	2	3	4	5	6	7	8		

1. (5+5+5=15 points) Evaluate the following limits, if they exist. Show your work. Do not use L'Hospital's rule.

(a) $\lim_{x\to 1}\frac{x^4-1}{x^8-1}$ If $x \neq 1$ then $\frac{x^4-1}{x^8-1} = \frac{x^4-1}{(x^4-1)(x^4+1)} = \frac{1}{x^4+1}$

So, $\lim_{x\to 1}\frac{x^4-1}{x^8-1} = \lim_{x\to 1}\frac{1}{x^4+1} = \frac{1}{2}$.

(b) $\lim_{x\to 0}\frac{\sin(|x|)}{x}$ $\lim_{x\to 0^-}\frac{\sin|x|}{x} = \lim_{x\to 0^-}\frac{\sin(-x)}{x} = -\lim_{x\to 0^-}\frac{\sin(x)}{x} = -1$,

$\lim_{x\to 0^+}\frac{\sin|x|}{x} = \lim_{x\to 0^+}\frac{\sin(x)}{x} = 1$. Thus the limit does not exist.

(c) $\lim_{x\to 0}(x^4+x^2)\sin(\frac{1}{x})$ Since $x^4+x^2 \geq 0$ and $|\sin(\frac{1}{x})| \leq 1$, it follows that $-x^4-x^2 \leq (x^4+x^2)\sin(\frac{1}{x}) \leq x^4+x^2$.

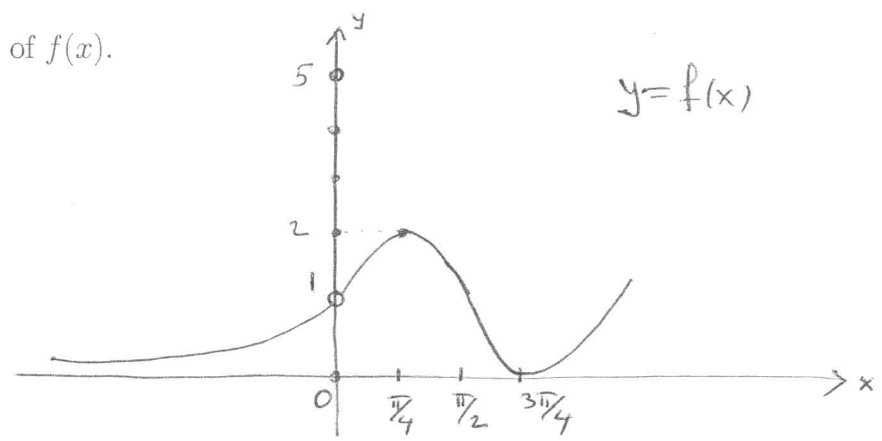
But $\lim_{x\to 0} \pm(x^4+x^2) = 0$. Hence

$\lim_{x\to 0}(x^4+x^2)\sin(\frac{1}{x}) = 0$ thanks to the Squeeze Theorem.

2. (5+5+5=15 points) Let $f(x)$ be the function given below:

$$f(x) = \begin{cases} \sin(2x)+1, & x>0, \\ -\frac{1}{x-1}, & x<0, \\ 5, & x=0 \end{cases}$$

(a) Sketch the graph of $f(x)$.



(b) Find all points a such that $\lim_{x\to a} f(x)$ exists.

Note that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{1-x} = 1$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (\sin(2x)+1) = 1$. Hence $\lim_{x\to 0} f(x)$ exists. Using the continuity of elementary functions, we conclude that $\lim_{x\to a} f(x)$ exists everywhere.

(c) Find all points where $f(x)$ is continuous.

$x=0$ is a removable discontinuity point.

The function $f(x)$ is continuous at all x, $x \neq 0$.

3. (10 points) Prove, using the precise definition of limit ($\epsilon - \delta$) that

$$\lim_{x\to 2} \frac{7}{5x-3} = 1$$

Take $\varepsilon > 0$. Then

$$\left|1 - \frac{7}{5x-3}\right| = \left|\frac{5x-10}{5x-3}\right| = \frac{5}{|5x-3|}|x-2|$$

One can assume that $|x-2| \leq 1$ ($\delta \leq 1$).

So, $-1 \leq x-2 \leq 1 \Rightarrow 1 \leq x \leq 3 \Rightarrow 5 \leq 5x \leq 15 \Rightarrow$

$\Rightarrow 2 \leq 5x-3 \leq 12 \Rightarrow \dfrac{5}{|5x-3|} \leq \dfrac{5}{2}$

Put $\delta = \min\{1, \frac{2\varepsilon}{5}\}$. Then

$$\left|1 - \frac{7}{5x-3}\right| = \frac{5}{|5x-3|}|x-2| \leq \frac{5}{2}\delta \leq \varepsilon$$

whenever $|x-2| \leq \delta$. Hence

$$\lim_{x\to 2} \frac{7}{5x-3} = 1.$$

4. (5+5+5=15 points) Compute the indicated derivatives. Do not simplify your answers. The notation $f^{(n)}(x)$ in part (c) means the n'th derivative.

(a) $(\sqrt[3]{\cos(x^3)})' = \frac{1}{3}(\cos(x^3))^{-2/3} \cdot (-\sin(x^3)) \cdot 3x^2$

(b) $\left(\frac{x \sin x}{x^2+1}\right)' = \frac{(\sin(x) + x\cos(x))(x^2+1) - 2x^2\sin(x)}{(x^2+1)^2}$

(c) $(x \cos x)^{(8)}$. $(x\cos(x))' = \cos(x) - x\sin(x)$,

$(x\cos(x))'' = \cancel{2}\sin(x) - x\cos(x)$

$(x\cos(x))^{(3)} = -2\cos(x) - \cos(x) + x\sin(x) =$

$= -3\cos(x) + x\sin(x)$

$(x\cos(x))^{(4)} = 4\sin(x) + x\cos(x)$

$(x\cos(x))^{(5)} = 4\cos(x) + \cos(x) - x\sin(x) =$

$= 5\cos(x) - x\sin(x)$

⋮

$(x\cos(x))^{(8)} = 8\sin(x) + x\cos(x)$

5. (12 points) Let $f(x) = 3x^3 - 20x + 12$. Show, by using the Intermediate Value Theorem, that $f(x) = 0$ has 3 solutions. (You must check the requirements of the Intermediate Value Theorem for full credit.)

Since $f(x)$ is a polynomial, it is a continuous function on the whole real line.

$f(-10) < 0$, $f(0) = 12 > 0$. By IVT, we have a root in $(-10, 0)$.

$f(0) > 0$, $f(1) < 0$. By IVT, we have a root in $(0, 1)$.

Finally, $f(1) < 0$, $f(10) > 0$ $\Rightarrow$ we have a root in $(1, 10)$ thanks to IVT.

So, we have 3 roots of the algebraic equation $f(x) = 0$.

6. (13 points) Consider the curve implicitly defined by the equation $\sin(y^2) = \cos(x^2)$. Find the equation of the tangent line to this curve at the point $(\sqrt{\frac{\pi}{6}}, \sqrt{\frac{\pi}{3}})$.

Assume $y = y(x)$ but locally. Then $\sin(y^2(x)) = \cos(x^2)$. Using the Chain Rule, we obtain that

$$2yy'\cos(y^2) = -\sin(x^2)\cdot 2\cdot x \Rightarrow$$

$$y' = \frac{-x\sin(x^2)}{y\cdot\cos(y^2)} \quad \text{whenever } y\cdot\cos(y^2) \neq 0.$$

The point belongs to the curve: $\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right)$.

The slope of the tangent line to the curve at $P(\sqrt{\pi/6}, \sqrt{\pi/3})$ is

$$y'\big|_P = -\frac{\sqrt{2}}{2}.$$

Finally

P-S-E: $y - \sqrt{\frac{\pi}{3}} = -\frac{\sqrt{2}}{2}\left(x - \sqrt{\frac{\pi}{6}}\right)$

is the equation of the tangent line.

7. (12 points) Suppose that Aylin and Burak are both making cylindrical pots from clay, and both cylinders are $1m$ tall with a radius of $3m$. Aylin increases only the radius of her pot, whereas Burak increases only the height of his pot, both at the same rate. Which of the two volumes increases faster? (Show your work.)

The volume of cylinder is $V = \pi r^2 h$.
So, $V(t) = \pi r^2(t)\, h(t)$ and the following rates are related:

$$V'(t) = 2\pi\, r(t)\, r'(t)\, h(t) + \pi r^2(t)\, h'(t).$$

In Aylin's case $h'(t) = 0$ and $r'(t) = a$, but in Burak's case $r'(t) = 0$ and $h'(t) = a$.
So,

Aylin	Burak
$V'(t) = 2\pi \cdot 3 \cdot a =$	$V'(t) = \pi \cdot 3^2 \cdot a =$
$= 6\pi \cdot a$	$= 9\pi a$

So, Burak's volume increases faster.

8. (8 points) Suppose that $f(x)$ is differentiable everywhere and that $(f \circ f)(x) = f(x)$ for all x. If c is a point such that $f(c) = c$, then show that $f'(c)$ must be 0 or 1.

Using the Chain Rule, we obtain that

$$f'(x) = f'(f(x)) \cdot f'(x), \quad \forall x \in \mathbb{R}.$$

In particular,

$$f'(c) = f'(f(c)) \cdot f'(c) = f'(c) \cdot f'(c) \quad \text{or}$$

$$f'(c)^2 = f'(c) \Rightarrow f'(c) = 0 \text{ or } f'(c) = 1.$$