	Calculus and Analytical Geometry Final	

Code : *Math 119* Acad.Year: *2009-2010* Semester : *Fall* Date : *13.1.2010* Time : *9:00* Duration : *165 minutes*	Last Name: Name : KEY Department: Signature: Student No.: Section: 7 QUESTIONS ON 8 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7		

Please show your work in all questions.

1. (4+4+4+4=16 points) Evaluate the following limits, if they exist.

(a) $\lim_{x\to\infty} e^x - x \; \{\infty - \infty\} = \lim_{x\to\infty} x\left(\frac{e^x}{x} - 1\right) = \infty$, for $\lim_{x\to\infty} \frac{e^x}{x} \overset{L.R.}{=} \infty$.

(b) $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\frac{1}{1+\cos 2x}} \; \{0^\infty\}$. If $y = (\sin x)^{(1+\cos(2x))^{-1}}$ then $\ln(y) = \frac{\ln(\sin(x))}{1+\cos(2x)} \left\{\frac{0}{0}\right\}$

$\lim_{x\to\pi/2} \ln(y) \overset{L.R.}{=} \lim_{x\to\pi/2} \frac{\cos(x)}{-2\sin(2x)\sin(x)} = \lim_{x\to\pi/2} \frac{\cancel{\cos(x)}}{-4\sin^2(x)\cancel{\cos(x)}} =$

$= \lim_{x\to\pi/2} \frac{1}{-4\sin^2(x)} = \frac{-1}{4}$

(c) $\lim_{x\to 0+} \frac{e^x}{x} = \infty$, for $\lim_{x\to 0^+} e^x = e^0 = 1$ ($y = e^x$ is a cont. function).

(d) Find the value of a such that the following limit is finite, and evaluate the limit:

$$\lim_{x\to 0} \frac{\cos(x^2) - 1 - ax^4}{x^8} \left\{\frac{0}{0}\right\} \overset{L.R.}{=} \lim_{x\to 0} \frac{-2x\sin(x^2) - 4ax^3}{8x^7}$$

$$= \lim_{x\to 0} \frac{\sin(x^2) - 2ax^2}{-4x^6} = \left\{\frac{0}{0}\right\} \overset{L.R.}{=} \lim_{x\to 0} \frac{2x\cos(x^2) - 4ax}{-4\cdot 6x^5} = \lim_{x\to 0} \frac{\cos(x^2) - 2a}{-12x^4}$$

If $a \neq \frac{1}{2}$ then $\lim_{x\to 0} (\cos(x^2) - 2a) = 1 - 2a \neq 0$, therefore the original limit can't be finite. Put $a = \frac{1}{2}$. Then $\lim_{x\to 0} \frac{\cos(x^2) - 2a}{-12x^4} = \left\{\frac{0}{0}\right\}$

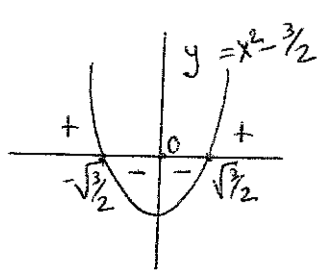
$= \lim_{x\to 0} \frac{-2x\sin(x^2)}{-12\cdot 4x^3} = \lim_{x\to 0} \frac{\sin(x^2)}{24x^2} = \frac{1}{24} < \infty$. So, the only possible choice for the number a is $\frac{1}{2}$.

2. (16 points) Sketch the graph of the function $f(x) = xe^{-x^2}$ indicating its domain, intercepts, symmetries, asymptotes, intervals of increase and decrease, critical points, local maxima and minima, concavity, and inflection points.

Note that $\mathrm{dom}(f) = \mathbb{R}$, $f(x) = 0 \Leftrightarrow x = 0$, $f(-x) = -f(x)$, and $\lim_{x\to\infty} f(x) = \left\{\frac{\infty}{\infty}\right\} \overset{L.R.}{=} \lim_{x\to\infty} \frac{1}{+2xe^{+x^2}} = 0$, that is, $y=0$ is a horizontal asymptote. Since $f'(x) = (1-2x^2)e^{-x^2}$, $C.P.(f) = \{\pm \frac{1}{\sqrt{2}}\}$. Further, $f''(x) = 2xe^{-x^2}(2x^2-3)$.

S.D.T. $f''\left(\frac{1}{\sqrt{2}}\right) < 0 \Rightarrow \frac{1}{\sqrt{2}}$ - loc. max.

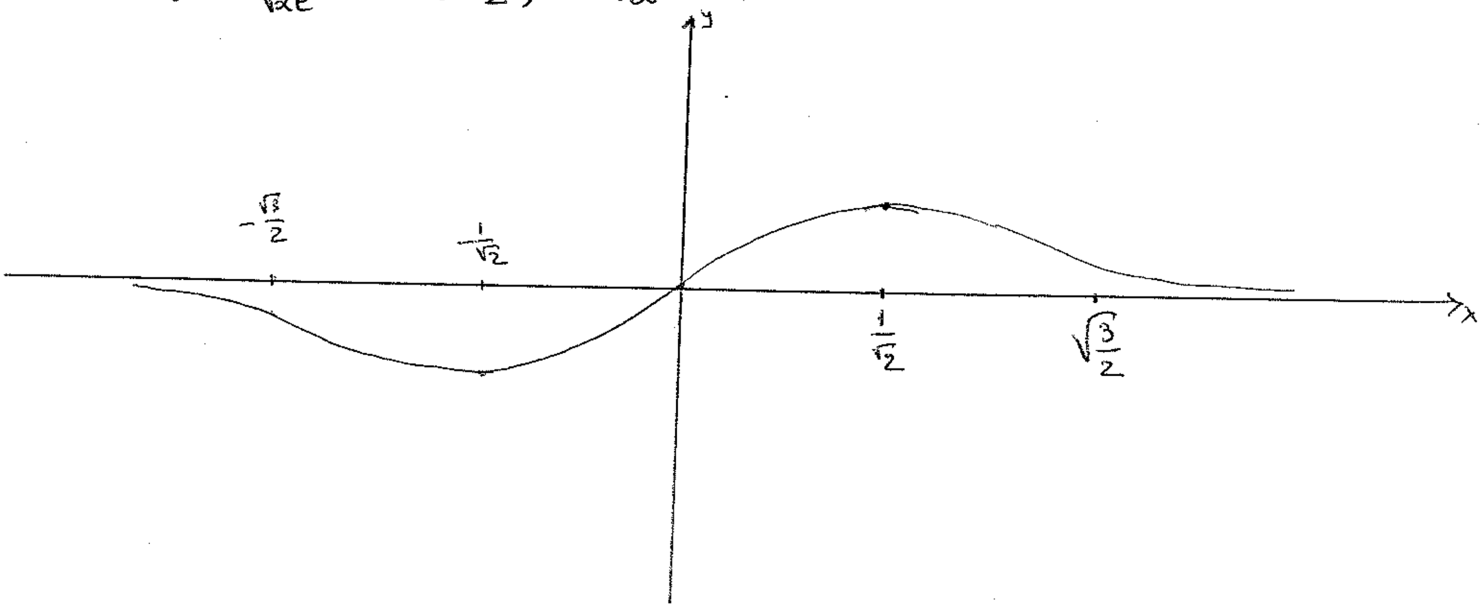
$f''\left(\frac{-1}{\sqrt{2}}\right) > 0 \Rightarrow \frac{-1}{\sqrt{2}}$ - loc. min.



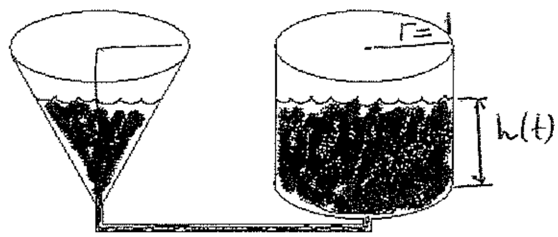
$C.P.(f') = \left\{-\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}\right\}$.

x	$x < -\sqrt{3/2}$	$-\sqrt{3/2} < x < 0$	$0 < x < \sqrt{3/2}$	$\sqrt{3/2} < x$
Sign $f''(x)$	−	+	−	+
Behavior $f(x)$	Conc. down	Conc. up	Conc. down	Conc. up

$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2e}}$, $f\left(\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}}\, e^{-3/2}$.



3. (10 points) A conical cup and a cylindrical cup are connected at their bottom ends (as shown in the figure) to share water. Water is pumped into the the cylindrical cup at the rate of 1 $\frac{\mathrm{m}^3}{\mathrm{s}}$. The radius of the cylinder is 1 m and the base radius of the cone is equal to its height. How fast is the level of water changing when it is 2 m deep?



If $h(t)$ is the height of the water level in the system at time t, and $V(t)$ is the relevant volume of water, then

$$V(t) = \pi r^2 h(t) + \frac{1}{3}\pi h^3(t),$$

where $r = 1$. So, the rates $V'(t)$ and $h'(t)$ are related. Namely,

$$V'(t) = \pi h'(t) + \pi h^2(t)\, h'(t) = \pi\left(1 + h^2(t)\right) h'(t).$$

We need to find $h'(t)$ whenever $h(t) = 2$. Since $V'(t) = \cancel{2}1 \ \frac{\mathrm{m}^3}{\mathrm{s}}$, it follows that

$$1 = \pi\,(1+4)\,h' \Rightarrow h' = \frac{1}{5\pi}.$$

4. (4+4+4+4+4+4=24 points) Evaluate the following integrals:

(a) $\int \frac{\sin(\ln x)}{x} dx. = \left| \begin{array}{l} u = \ln(x) \\ du = \frac{dx}{x} \end{array} \right| = \int \sin(u)\, du = -\cos(u) + C =$

$= -\cos(\ln(x)) + C$

(b) $\int_0^4 \frac{1}{(x-3)^2} dx.$ Since $f(x) = \frac{1}{(x-3)^2}$ is discontinuous at $x=3$,

we deal with the improper integral.

$\int_0^3 \frac{dx}{(x-3)^2} = \lim_{b \to 3^-} \int_0^b \frac{dx}{(x-3)^2} = \lim_{b \to 3^-} \left(\frac{1}{3-b} - \frac{1}{3} \right) = +\infty$

So, the improper integral diverges.

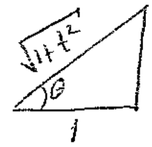
(c) $\int \tan^3 x \sec^3 x dx. = \int \tan^2(x) \sec^2(x) \tan(x) \sec(x)\, dx =$

$= \int (\sec^2(x) - 1) \sec^2(x) \tan(x) \sec(x) dx = \left| \begin{array}{l} u = \sec(x) \\ du = \tan(x)\sec(x) dx \end{array} \right|$

$= \int (u^2 - 1) u^2\, du = \frac{u^5}{5} - \frac{u^3}{3} + C =$

$= \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

(d) $\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{dx}{((x+1)^2+1)^2} = \left| \begin{array}{l} t = x+1 \\ dt = dx \end{array} \right| = \int \frac{dt}{(1+t^2)^2} = \left| \begin{array}{l} t = \tan(\theta) \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ dt = \sec^2(\theta)\,d\theta \\ 1+t^2 = \sec^2(\theta) \end{array} \right|$



$$= \int \frac{\sec^2(\theta)\,d\theta}{\sec^4(\theta)} = \int \cos^2(\theta)\,d\theta = \frac{1}{2}\int (1+\cos(2\theta))\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C = \frac{1}{2}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$

$$= \frac{1}{2}\left(\arctan(x+1) + \frac{x+1}{1+(x+1)^2}\right) + C$$

(e) $\int_4^9 \frac{\sqrt{x}}{x-1}\,dx = \left| \begin{array}{ll} u = \sqrt{x}, & x=4 \Rightarrow u=2 \\ du = \frac{dx}{2u}, & x=9 \Rightarrow u=3 \\ x = u^2 \Rightarrow x-1 = u^2-1 & \end{array} \right| = \int_2^3 \frac{2u^2\,du}{u^2-1} = 2 + 2\int_2^3 \frac{du}{u^2-1}$

$$= 2 + \int_2^3 \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du = 2 + \ln(2) + \ln(3) - \ln(4) =$$

$$= 2 + \ln(3) - \ln(2) = 2 + \ln\left(\frac{3}{2}\right)$$

(f) $\int \frac{\arctan x}{x^2}\,dx = -\int \arctan(x)\left(\frac{1}{x}\right)' dx = -\arctan(x)\cdot\frac{1}{x} +$

$$+ \int \frac{1}{x}\arctan'(x)\,dx = -\frac{\arctan(x)}{x} + \int \frac{dx}{x(1+x^2)}$$

Note that $\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2} \Rightarrow \int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} -$

$- \int \frac{x\,dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln(x^2+1) + C$ Hence

$$\int \frac{\arctan(x)}{x^2}\,dx = -\frac{\arctan(x)}{x} + \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

5. (8 points) Find the arclength of the curve

$$y = \ln(\cos x), \quad 0 \leq x \leq \pi/3$$

If $f(x) = \ln(\cos(x))$, $0 \leq x \leq \pi/3$, then

$$\text{Arclength}(f) = \int_0^{\pi/3} (1 + f'(x)^2)^{1/2}\, dx =$$

$$= \int_0^{\pi/3} (1 + \tan^2(x))^{1/2}\, dx = \int_0^{\pi/3} |\sec(x)|\, dx$$

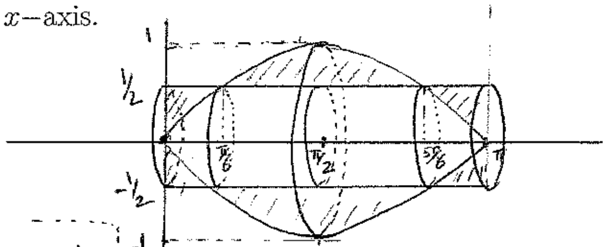
($\sec(x) > 0$ whenever $0 \leq x \leq \pi/3$)

$$= \int_0^{\pi/3} \sec(x)\, dx = \ln|\sec(x) + \tan(x)| \Big|_0^{\pi/3}$$

$$= \ln(2 + \sqrt{3}).$$

6. (8+8=16 points) Consider the region bounded by the curves $y = \sin x$, $y = \frac{1}{2}$, $x = 0$ and $x = \pi$ (This region also appeared in one question in Midterm 2). Express as integrals, **but do not evaluate**, the volumes described below: *(To receive full credit, your answers must not involve absolute value signs.)*

(a) The volume obtained by rotating the region about the x–axis.



$$V = 2\int_0^{\pi/6} \left(\pi \frac{1}{2^2} - \pi \sin^2(x)\right)dx + \int_{\pi/6}^{5\pi/6} \left(\pi \sin^2(x) - \pi \frac{1}{2^2}\right)dx$$

(b) The volume obtained by rotating the region about the line $x = -2$.

$$V = \int_0^{\pi/6} 2\pi(x+2)\cdot\left(\frac{1}{2} - \sin(x)\right)dx + \int_{\pi/6}^{5\pi/6} 2\pi(x+2)\left(\sin(x) - \frac{1}{2}\right)dx$$

$$+ \int_{5\pi/6}^{\pi} 2\pi(x+2)\left(\frac{1}{2} - \sin(x)\right)dx$$

7. (10 points) Suppose that $f(x)$ is a continuous function such that $f(0) = 5$, $f(1) = 0$, $f'(x) < 0$ for all $x \in (0,1)$, and the average value of $f(x)$ on $[0,1]$ is 3. Find the average value of $f^{-1}(x)$ on $[0,5]$. *(Hint: Substitute $u = f^{-1}(x)$ in the relevant integral.)*

We know that $\int_0^1 f(x)\,dx = 3$. Then

$$\frac{1}{5}\int_0^5 f^{-1}(x)\,dx = \left| \begin{array}{l} u = f^{-1}(x),\ x = f(u) \\ dx = f'(u)\,du \\ x=0 \Rightarrow u = f^{-1}(0) = 1 \\ x = 5 \Rightarrow u = f^{-1}(5) = 0 \end{array} \right| = \frac{1}{5}\int_1^0 u\, f'(u)\,du$$

$$= \frac{-1}{5}\int_0^1 u\, f'(u)\,du = \frac{-1}{5}\left(u f(u)\Big|_0^1 - \int_0^1 f(u)\,du \right)$$

$$= -\frac{1}{5}\left(f(1) - 3\right) = -\frac{1}{5}(0-3) = \frac{3}{5}, \text{ that is,}$$

$$\text{Average value of } f^{-1} = \frac{1}{5}\int_0^5 f^{-1}(x)\,dx = \frac{3}{5}.$$