

Original article

# Bifurcation of discontinuous limit cycles of the Van der Pol equation

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## Abstract

In this paper, we apply the methods of  $B$ -equivalence and  $\psi$ -substitution to prove the existence of discontinuous limit cycle for the Van der Pol equation with impacts on surfaces. The result is extended through the center manifold theory for coupled oscillators. The main novelty of the result is that the surfaces, where the jumps occur, are not flat. Examples and simulations are provided to demonstrate the theoretical results as well as application opportunities.

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## 1. Introduction and preliminaries

Getting bifurcation in dynamics with impacts relies mainly on collisions near the impact point(s). That is why they are called corner-collision, border-collision, crossing-sliding, grazing-sliding, switching-sliding, etc., bifurcations [9,12,13,15,17,22,23,30]. That is, the bifurcations are located geometrically. In our present result, we do not have the geometrical source of bifurcation. It is rather reasoned by specifically arranged interaction of continuous and discontinuous stages of the process. To be precise, we use a generalized eigenvalue to evaluate which we apply a characteristic of the impact as well as of the continuous process between moments of discontinuity. This approach when continuous and discontinuous stages are equally participated in creating a certain phenomena is common for the theory of differential equations with impulses [3,34]. Our results are, rather, close to those, which obtained for systems where continuous flows and surfaces of discontinuity are transversal [2,3,5,14,24].

The main instruments in our paper, except for the Hopf bifurcation technique, are the methods of  $B$ -equivalence and  $\psi$ -substitution developed in our papers [1–4,6] for discontinuous limit cycles, and one has to emphasize that the set of all periodic solutions of the non-perturbed system is a proper subset of all solutions near the origin. By a discontinuous cycle, we mean a trajectory of a discontinuous periodic solution.

The Van der Pol equation arises in the study of circuits containing vacuum tubes and is given by

$$y'' + \varepsilon(1 - y^2)y' + y = 0 \quad (1)$$

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where  $\varepsilon$  is a real parameter. If  $\varepsilon = 0$ , the equation reduces to the equation of simple harmonic motion  $y'' + y = 0$ . The term  $\varepsilon(1 - y^2)y'$  in (1) is usually regarded as the friction or resistance. If the coefficient  $\varepsilon(1 - y^2)$  is positive, then we have the case of “positive resistance”, and when the coefficient  $\varepsilon(1 - y^2)$  is negative then we have the case of “negative resistance”. This equation, introduced by Lord Rayleigh (1896), was studied by Van der Pol (1927) [36] both theoretically and experimentally using electric circuits.

Hopf bifurcation is an attractive subject of analysis for mathematicians as well as for mechanics and engineers [2,5,8,11,13–15,22,25,26,28,31,35,36]. Many papers and books have been published about mechanical and electrical systems with impacts [7,9,12,17,20,27,30,37].

We consider the model with impulses on surfaces which are places in the phase space and are essentially nonlinear while it is known that the Hopf bifurcation is considered either with linear surfaces of discontinuity or with fixed moments of impulses [7–10,13,15,17,18,22,33]. We have developed a special effective approach to analyze the problem in depth which consists of the method of reduction of equations with variable moments of impacts to systems with fixed moments of impacts [3], a class of equations on variable time scales [2,6], a transformation of equations on time scales to systems with impulses [4]. This is all the theoretical basis of the present results.

Specifically, we consider the following system:

$$\begin{aligned} y'' + 2\alpha y' + (\alpha^2 + \beta^2)y &= F(y, y', \mu), \quad (y, y') \notin \Gamma(\mu), \\ \Delta y'|_{(y,y') \in \Gamma(\mu)} &= cy + dy' + J(y, y', \mu), \end{aligned} \quad (2)$$

where  $\alpha, \beta \neq 0, c, d$  are real constants with  $c = \alpha d$ ,  $F$  and  $J$  are analytic functions in all variables.  $\Gamma(\mu)$  is the set of discontinuity whose equation is given by  $m_1 y + m_2 y' + \tau(y, y', \mu) = 0, y > 0$ , for some real numbers  $m_1, m_2$ , and the function  $\tau(y, y', \mu)$  stands for a small perturbation,  $\Delta y'|_{(y,y') \in \Gamma(\mu)} = y'(\theta^+) - y'(\theta)$  denotes the jump operator in which  $\theta$  is the time when the solution  $(y, y')$  meets the discontinuity set  $\Gamma(\mu)$ , that is,  $\theta$  is such that  $m_1 y(\theta) + m_2 y'(\theta) + \tau(y(\theta), y'(\theta), \mu) = 0$ , and  $y'(\theta^+)$  is the right limit of  $y'(t)$  at  $t = \theta$ . After the impact, the phase point  $(y(\theta^+), y'(\theta^+))$  will belong to the set  $\Gamma'(\mu) = \{(u, v) \in \mathbb{R}^2 : u = y, v = cy + (1 + d)y' + J(y, y', \mu), (y, y') \in \Gamma(\mu)\}$ . Here  $y(\theta^+)$  is the right limit of  $y(t)$  at  $t = \theta$ . One can easily see that nonlinearity is inserted into all parts of the model including the surface of discontinuity.

If we choose  $\alpha = \varepsilon/2, \beta = \sqrt{1 - \alpha^2}$  and  $F(y, y', \mu) = \varepsilon y^2 y'$  in the differential equation of the system (2), then the Van der Pol equation will be obtained. Therefore, (1) is a special case of (2), if the impulsive condition is not considered. Note that if  $F(y, y', \mu) = \varepsilon_2 y^2 y'$  for some nonzero constant  $\varepsilon_2$ , we still have (1) after using the linear transformation  $y = \sqrt{\varepsilon/\varepsilon_2} z$  of the dependent variable.

To explain our application motivations, we consider the oscillator which is subdued to the impacts modeled by the Newton's law of restitution as a concrete mechanical problem. Consider the system

$$\begin{aligned} y'' + \varepsilon_1 y' + y &= \varepsilon_2 y^2 y', \quad (y, y') \notin \Gamma, \\ \Delta y'|_{(y,y') \in \Gamma} &= dy', \end{aligned} \quad (3)$$

where  $\varepsilon_1, \varepsilon_2$  are constants,  $d = e^{2\pi\varepsilon_1(4 - \varepsilon_1^2)^{-1/2}} - 1$ ,  $\Gamma$  is the half line  $y = 0, y' > 0$ . As mentioned above, the last system is a generalization of the Van der Pol equation with impacts of Newton's type. If one takes (3) with  $\varepsilon_2 = 0$ , then the system is

$$\begin{aligned} y'' + \varepsilon_1 y' + y &= 0, \quad (y, y') \notin \Gamma, \\ \Delta y'|_{(y,y') \in \Gamma} &= dy'. \end{aligned} \quad (4)$$

Note that the general solution of the differential equation without impulse condition in (4) is given by

$$y(t) = e^{-\varepsilon_1 t/2} \left( C_1 \cos \left( \frac{(4 - \varepsilon_1^2)^{1/2} t}{2} \right) + C_2 \sin((4 - \varepsilon_1^2)^{1/2} t/2) \right), \quad (5)$$

where  $C_1$  and  $C_2$  are arbitrary real constants. Let  $(0, y'_0)$  be any point on the line  $\Gamma' = \Gamma$ . That is, assume that  $y'_0 > 0$ . Then  $y(0) = 0, y'(0) = y'_0$  in (5) gives us  $C_1 = 0, C_2 = 2y'_0(4 - \varepsilon_1^2)^{-1/2}$ . Thus, we obtain

$$y(t) = 2y'_0(4 - \varepsilon_1^2)^{-1/2} e^{-\varepsilon_1 t/2} \sin((4 - \varepsilon_1^2)^{1/2} t/2).$$

Now, the first impact action takes place at time  $t = T$  where  $T > 0$  and  $y(T) = 0$ , which means  $T = 4\pi(4 - \varepsilon_1^2)^{-1/2}$ . At that time, we obtain that  $y'(T) = e^{-2\pi\varepsilon_1(4 - \varepsilon_1)^{-1/2}} y'_0$ , and after the impact we have  $y'(T^+) = (1 + d)y(T) = y'_0$ . Therefore, all solutions starting on  $\Gamma'$  are  $T = 4\pi(4 - \varepsilon_1^2)^{-1/2}$  periodic. One such solution with  $y'_0 = 0.06$  is depicted in Fig. 3.

The obtained result for (4) shows that the origin is the center for the system, and it is analogous of the planar degenerated linear homogeneous system with constant coefficients in the original Hopf bifurcation theorem. This gives us a hint to apply the bifurcation technique to more general systems of type (2).

Systems of type (3) has been analyzed in many papers and books [7,8,13,14,25,26,34,36] and references cited there. Here, we have mentioned just some of them. We will apply the results of our paper to prove the existence of a stable periodic motion of the model, in the perturbed system corresponding to this model. Moreover, in Example 3.1 we will handle a more complicated case of two coupled oscillators where one of the oscillators is subdued to the impacts modeled by the Newton’s law of restitution.

We strictly believe that results of the present paper can be applied to other mechanical, electrical as well as biological problems if one adopts the models by special transformations to the considered case. Moreover, in the upcoming researches, we plan to weaken some restrictions on the model. For example, the approach can be extended to equations where surfaces of discontinuity do not intersect at the origin.

Finally, in the present study, we extend results to the two-oscillators model through the application of center manifold. The analysis developed in our paper can be applied to various problems of mechanics, electronics and biology.

## 2. Theoretical results

### 2.1. Reduction to polar coordinates

Assume that functions  $F, J$  and  $\tau$  are analytic in all variables,

$$F(0, 0, \mu) = J(0, 0, \mu) = \tau(0, 0, \mu) = 0$$

for all  $\mu$ , and first derivatives of  $F, J$  and  $\tau$  at  $(y, y', \mu) = (0, 0, 0)$  vanish. We start the theoretical investigation by writing (2) in the following form:

$$\begin{aligned} y'' + 2\alpha(\mu)y' + (\alpha^2(\mu) + \beta^2(\mu))y &= G(y, y', \mu), \quad (y, y') \notin \Gamma(\mu), \\ \Delta y'|_{(y,y') \in \Gamma(\mu)} &= cy + dy' + J(y, y', \mu), \end{aligned} \tag{6}$$

where  $\alpha(\mu) = \alpha - (1/2)(\partial F/\partial y')(0, 0, \mu)$ ,  $\beta(\mu) = (\alpha^2 + \beta^2 - \alpha^2(\mu) - (\partial F/\partial y')(0, 0, \mu))^{1/2}$ ,  $G(y, y', \mu) = F(y, y', \mu) - y(\partial F/\partial y)(0, 0, \mu) - y'(\partial F/\partial y')(0, 0, \mu)$ . Note that the functions  $G, (\partial G/\partial y)$  and  $(\partial G/\partial y')$  vanish at  $(0, 0, \mu)$  for all  $\mu$ .  $\Gamma(\mu)$  can also be written as

$$\Gamma(\mu) : m_1(\mu)y + m_2(\mu)y' + \tau_2(y, y', \mu) = 0,$$

where  $m_1(\mu) = m_1 + \partial\tau/\partial y(0, 0, \mu)$ ,  $m_2(\mu) = m_1 + \partial\tau/\partial y'(0, 0, \mu)$  and  $\tau_2(y, y', \mu) = \tau(y, y', \mu) - y\partial\tau/\partial y(0, 0, \mu) - y'\partial\tau/\partial y'(0, 0, \mu)$ .

We write (6) as a system of first order equations in  $x_1x_2$ -plane so that the linear part has the coefficient matrix in Jordan form. For this purpose, we let  $x_1 = (\alpha(\mu)y + y')/\beta(\mu)$  and  $x_2 = y$ . Then (6) is written as

$$\begin{aligned} x'_1 &= -\alpha(\mu)x_1 - \beta(\mu)x_2 + H(x_1, x_2, \mu), \\ x'_2 &= \beta(\mu)x_1 - \alpha(\mu)x_2, \quad (x_1, x_2) \notin \Gamma(\mu), \\ \Delta x_1|_{(x_1,x_2) \in \Gamma(\mu)} &= Ix_1 + K(x_1, x_2, \mu), \end{aligned} \tag{7}$$

where  $I = d$ , functions  $H$  and  $K$  are analytic in all their variables and they carry all the properties of  $G$  and  $J$ , respectively. The discontinuity surface  $\Gamma(\mu)$  is given by

$$\Gamma(\mu) : m_2(\mu)\beta(\mu)x_1 + (m_1(\mu) - \alpha(\mu)m_2(\mu))x_2 + \tau_3(x_1, x_2, \mu) = 0.$$

Note that system (7) is more convenient to use the polar coordinates.

We shall now introduce the polar coordinates, but first, consider the set of discontinuity points  $\Gamma(\mu)$  in polar coordinates. Using the change of variables  $x_1 = r \cos \phi$ ,  $x_2 = r \sin \phi$ , the curve  $\Gamma(\mu)$  is represented as

$$\Gamma(\mu) : \phi = \phi_0(\mu) + v(r, \phi, \mu),$$

where  $\phi_0(\mu) = \arctan(m_2(\mu)\beta(\mu)/(\alpha(\mu)m_2(\mu) - m_1(\mu)))$ ,  $v$  is analytic in all variables,  $2\pi$ -periodic in  $\phi$  and  $v = \mathcal{O}(r)$ . Thus, using the polar coordinates, (7) is transformed into the system

$$\begin{aligned} r' &= -\alpha(\mu)r + R_1(r, \phi, \mu), \\ \phi' &= \beta(\mu) + R_2(r, \phi, \mu), \quad (r, \phi) \notin \Gamma(\mu), \\ \Delta r|_{(r,\phi) \in \Gamma(\mu)} &= k(\mu)r + R(r, \phi, \mu), \\ \Delta \phi|_{(r,\phi) \in \Gamma(\mu)} &= -\theta(\mu) + \Theta(r, \phi, \mu), \end{aligned} \tag{8}$$

where  $R_1, R_2, R, \Theta$  are all  $2\pi$ -periodic in  $\phi$ ,  $R_1 = \mathcal{O}(r^2)$ ,  $R_2 = \mathcal{O}(r)$ ,  $R = \mathcal{O}(r^2)$ ,  $\Theta = \mathcal{O}(r)$  and

$$k(\mu) = \sqrt{(I^2 + 2I)\cos^2(\phi_0(\mu)) + 1} - 1.$$

Eliminating the time variable, and considering  $\phi$  as the independent variable, we can write (8) as

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda(\mu)r + \mathcal{R}(r, \phi, \mu), \quad \phi \neq \phi_0(\mu) + \xi_{2j}(r, \phi, \mu), \\ \Delta r|_{\phi=\phi_0(\mu)+\xi_{2j}(r,\phi,\mu)} &= k(\mu)r + R(r, \phi, \mu), \\ \Delta \phi|_{\phi=\phi_0(\mu)+\xi_{2j}(r,\phi,\mu)} &= -\theta(\mu) + \Theta(r, \phi, \mu), \end{aligned} \tag{9}$$

where  $\lambda(\mu) = -\alpha(\mu)/\beta(\mu)$ ,  $\xi_{2j}(r, \phi, \mu) = v(r, \phi, \mu)$ . In a neighborhood of the origin, it is easily seen that, all solutions of (9), except for the trivial solution, rotate around the origin. Note that the impacts occur once in every two meetings of the trajectory with the discontinuity set  $\Gamma(\mu)$ . To indicate this notion we use  $2j$  in the subscript in (9).

During one rotation around the origin, if a solution  $r(\phi)$  of (9) performs the first impact at the moment when  $\phi = \eta_{2j}$ , that is,  $\eta_{2j} = \phi_0 + \xi_{2j}(r(\eta_{2j}), \eta_{2j}, \mu)$  and if it jumps to the point  $(r(\gamma_{2j}), \gamma_{2j})$  after the impact, where  $\gamma_{2j} = \eta_{2j} - \theta(\mu) + \Theta(r(\eta_{2j}), \eta_{2j}, \mu)$ , then this solution is defined on the variable time scale  $\cup_{j \in \mathbb{Z}} (\gamma_{2j} + 2j\pi, \eta_{2j} + 2(j+1)\pi)$ . The variable time scale depends on the initial data and the time scale is different for different solution. Thus (9) is considered as an impulsive differential equation on variable time scale [6].

## 2.2. The B-equivalent system

In this part, we shall reduce the system in polar coordinates on variable time scale (9) to the system on the non variable time scale with transition condition [4], using the method of B-equivalence [2,3,6].

Let  $r(\phi)$  be a solution of (9) with initial condition  $r(\phi_0(\mu)) = r$ , and assume that  $\phi = \eta_{2j}$  is the first from left solution of  $\phi = \phi_0(\mu) + \xi_{2j}(r, \phi, \mu)$ . That is, assume that  $r(\phi)$  performs the first impact at the moment when  $\phi = \eta_{2j}$ . Let the solution  $r(\phi)$  jump to the point  $(r(\gamma_{2j}), \gamma_{2j})$  after the impact. Then we have

$$\begin{aligned} \gamma_{2j} &= \eta_{2j} - \theta(\mu) + \Theta(r(\eta_{2j}), \eta_{2j}, \mu), \\ r(\gamma_{2j}) &= (1 + k(\mu))r(\eta_{2j}) + R(r(\eta_{2j}), \eta_{2j}, \mu). \end{aligned}$$

Throughout the paper,  $\widehat{[a, b]}$  denotes the oriented interval for any  $a, b \in \mathbb{R}$ . That is, it denotes  $[a, b]$  when  $a < b$ , and it denotes  $[b, a]$  otherwise.

Denote by  $r_1(\phi)$  the solution of

$$\frac{dr}{d\phi} = \lambda(\mu)r + \mathcal{R}(r, \phi, \mu) \tag{10}$$

with the initial condition  $r_1(\gamma_{2j}) = r(\gamma_{2j})$ .

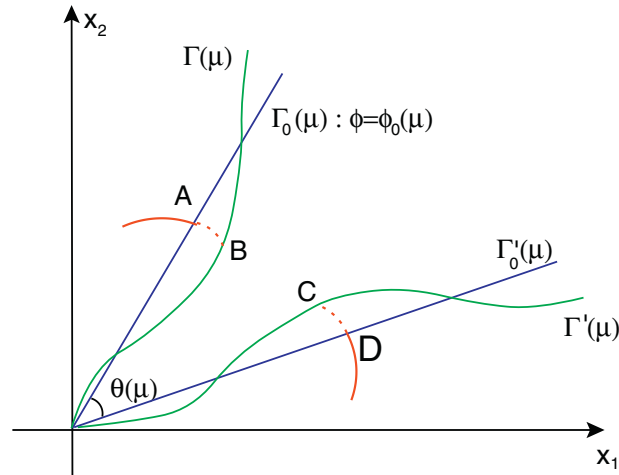


Fig. 1. B-equivalence and the map W.

For  $\phi \in [\widehat{\phi_0(\mu)}, \widehat{\eta_{2j}}]$ , we have

$$r(\phi) = r + \int_{\phi_0(\mu)}^{\phi} [\lambda(\mu)r(s) + \mathcal{R}(r(s), s, \mu)]ds,$$

and on the interval  $[\gamma_{2j}, \widehat{\phi_0(\mu)} - \theta(\mu)]$ , we have

$$r_1(\phi) = r(\gamma_{2j}) + \int_{\gamma_{2j}}^{\phi} [\lambda(\mu)r_1(s) + \mathcal{R}(r_1(s), s, \mu)]ds.$$

Thus,

$$\begin{aligned} r_1(\phi_0(\mu) - \theta(\mu)) &= r(\gamma_{2j}) + \int_{\gamma_{2j}}^{\phi_0(\mu) - \theta(\mu)} [\lambda(\mu)r_1(s) + \mathcal{R}(r_1(s), s, \mu)]ds = (1 + k(\mu))r(\eta_{2j}) + \mathcal{R}(r(\eta_{2j}), \eta_{2j}, \mu) \\ &+ \int_{\gamma_{2j}}^{\phi_0(\mu) - \theta(\mu)} [\lambda(\mu)r_1(s) + \mathcal{R}(r_1(s), s, \mu)]ds = (1 + k(\mu)) \\ &\times \left[ r + \int_{\phi_0(\mu)}^{\eta_{2j}} [\lambda(\mu)r(s) + \mathcal{R}(r(s), s, \mu)]ds \right] + \mathcal{R}(r(\eta_{2j}), \eta_{2j}, \mu) \\ &+ \int_{\gamma_{2j}}^{\phi_0(\mu) - \theta(\mu)} [\lambda(\mu)r_1(s) + \mathcal{R}(r_1(s), s, \mu)]ds. \end{aligned}$$

We now let

$$\begin{aligned} W(r, \mu) &= r_1(\phi_0(\mu) - \theta(\mu)) - (1 + k(\mu))r = (1 + k(\mu)) \int_{\phi_0(\mu)}^{\eta_{2j}} [\lambda(\mu)r(s) + \mathcal{R}(r(s), s, \mu)]ds \\ &+ \mathcal{R}(r(\eta_{2j}), \eta_{2j}, \mu) + \int_{\gamma_{2j}}^{\phi_0(\mu) - \theta(\mu)} [\lambda(\mu)r_1(s) + \mathcal{R}(r_1(s), s, \mu)]ds. \end{aligned}$$

Defining  $W(u, \mu)$  and  $W(v, \mu)$ , using the smallness of the right side function  $\mathcal{R}$  and continuous dependence on initial data, we can show that there exists a Lipschitz constant  $\ell$  and a bounded function  $m(\ell)$  such that

$$|W(u, \mu) - W(v, \mu)| \leq \ell m(\ell)|u - v| \tag{11}$$

for all  $u, v \in \mathbb{R}$ . In Fig. 1, the construction of  $W$  is demonstrated. There, the point  $A$  is  $(r, \phi_0(\mu))$ ,  $B$  is the point where the first impact occurs. That is,  $B$  is the point  $(r(\eta_{2j}), \eta_{2j})$ . The phase point jumps to  $C$  after the impact. That is,  $C$  is the point  $(r(\gamma_{2j}), \gamma_{2j})$ . Finally,  $D$  is the point  $(r_1(\phi_0(\mu) - \theta(\mu)), \phi_0(\mu) - \theta(\mu))$ .

Now, we define the following system

$$\begin{aligned}\frac{d\rho}{d\phi} &= \lambda(\mu)\rho + \mathcal{R}(\rho, \phi, \mu), \quad \phi \neq \phi_0(\mu), \\ \Delta\rho|_{\phi=\phi_0(\mu)} &= k(\mu)\rho + W(\rho, \mu), \\ \Delta\phi|_{\phi=\phi_0(\mu)} &= -\theta(\mu).\end{aligned}\tag{12}$$

It can be seen that in a neighborhood of the origin, all solutions of (12), except for the trivial solution, rotate around the origin, as in the case of (9). The variable  $\phi$  ranges over the time scale  $\cup_{n \in \mathbb{Z}} (\phi_0(\mu) - \theta(\mu) + 2n\pi, \phi_0(\mu) + 2(n+1)\pi]$ . It can be seen that the time scale is a union of overlapping intervals. Indeed, (12) is an example of differential equation on a time scale with transition condition (DETCV) [4]. Nevertheless, the time scale is of a new type, since the intervals are overlapping.

**Definition 2.1.** [2] We say that (9) and (12) are  $B$ -equivalent in a neighborhood of the origin if corresponding to each solution  $r(\phi)$  of (9), there is a solution  $\rho(\phi)$  of (12) such that  $\rho(\phi) = r(\phi)$  for all  $\phi$  except possibly on the intervals  $[\phi_0(\mu), \eta_{2j}]$  and  $[\gamma_{2j}, \phi_0(\mu) - \theta(\mu)]$ , where  $\eta_{2j} = \eta_{2j}(\mu)$  is the angle when  $r(\phi)$  meets  $\Gamma(\mu)$  and  $\gamma_{2j} = \gamma_{2j}(\mu)$  is the angle where  $r(\phi)$  is after the impact.

From the construction made above for  $W$ , one can easily see that the following lemma is valid.

**Lemma 2.1.** Systems (9) and (12) are  $B$ -equivalent in a neighborhood of the origin.

### 2.3. $\psi$ -Substitution

The independent variable  $\phi$  in (12) ranges over the domain  $\cup_{n \in \mathbb{Z}} I_n$  where

$$I_n = (\phi_0(\mu) - \theta(\mu) + 2n\pi, \phi_0(\mu) + 2(n+1)\pi].$$

Note that  $\cup_{n \in \mathbb{Z}} I_n$  is the union of overlapping closed intervals. That is,

$$I_{n,n+1} := I_n \cap I_{n+1} = (\phi_0(\mu) - \theta(\mu) + 2(n+1)\pi, \phi_0(\mu) + 2(n+1)\pi].$$

Therefore, we use a generalization of the so-called  $\psi$ -substitution [2,4]. In our case, the  $\psi$ -substitution is defined to be the shifting of the intervals. More precisely, we redefine the intervals  $I_n$  as  $I'_n := I_n + n\theta(\mu)$ . The piece of graph of a solution is shifted accordingly. After the  $\psi$ -substitution we obtain  $I'_n \cap I'_{n+1} = \{\}$ , and hence the graph of any trajectory is a single-valued function.

Thus, using the  $\psi$ -substitution, we write (12) as

$$\begin{aligned}\frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + \tilde{\mathcal{R}}(\rho, \varphi, \mu), \quad \varphi \neq \phi_0(\mu), \\ \Delta\rho|_{\varphi=\phi_0(\mu)} &= k(\mu)\rho + \tilde{W}(\rho, \mu),\end{aligned}\tag{13}$$

where  $\varphi = \psi(\phi)$ . To investigate the Hopf-bifurcation in (13), we follow the classical method, and assume that the non-perturbed system has a family of periodic solutions and the origin in the perturbed system corresponding to  $\mu = 0$  is asymptotically stable. Consider the case  $\mu = 0$ :

$$\begin{aligned}\frac{d\rho}{d\varphi} &= \lambda\rho + \tilde{\mathcal{R}}(\rho, \varphi), \quad \varphi \neq \phi_0, \\ \Delta\rho|_{\varphi=\phi_0} &= k\rho + \tilde{W}(\rho),\end{aligned}\tag{14}$$

where  $\lambda, k, \phi_0, \tilde{\mathcal{R}}(\rho, \varphi)$  and  $\tilde{W}(\rho)$  are the values of  $\lambda(\mu), k(\mu), \phi_0(\mu), \tilde{\mathcal{R}}(\rho, \varphi, \mu)$  and  $\tilde{W}(\rho, \mu)$  at  $\mu = 0$ , respectively. The non-perturbed system corresponding to (14) is

$$\begin{aligned}\frac{d\rho}{d\varphi} &= \lambda\rho, \quad \varphi \neq \phi_0, \\ \Delta\rho|_{\varphi=\phi_0} &= k\rho.\end{aligned}\tag{15}$$

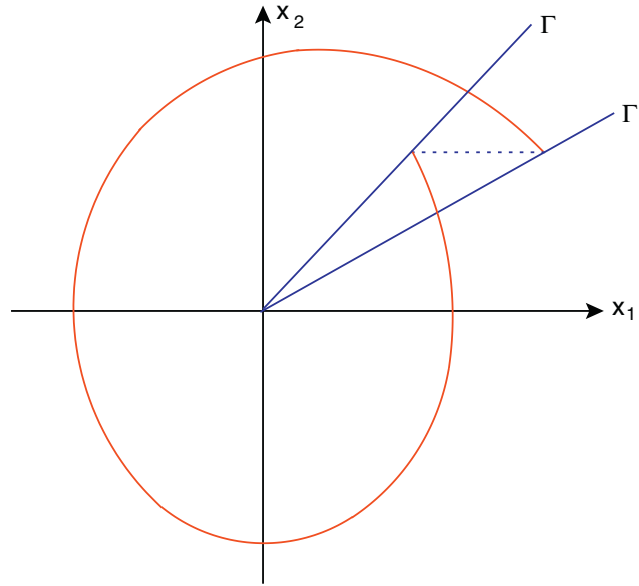


Fig. 2. All solutions of (15) with the initial condition on  $\Gamma'$  are periodic.

The impacts in (15) occur on the line  $\Gamma : \phi = \phi_0$ , and, after the impact, the trajectory is on the line  $\Gamma' : \phi = \phi_0 - \theta$ . The solution  $r(\phi) = r(\phi, \phi_0 - \theta, r_0)$ ,  $r_0 > 0$ , of (15) with the initial condition  $r(\phi_0 - \theta) = r_0$ , where the point  $(r_0, \phi_0 - \theta)$  is on the line  $\Gamma'$  (see Fig. 2), is given by  $r(\phi) = r_0 e^{\lambda(\phi - \phi_0 + \theta)}$  for  $\phi_0 - \theta \leq \phi \leq \phi_0 + 2\pi$ . Therefore, before the first impact, we have  $r(\phi_0 + 2\pi) = r_0 e^{-\alpha T}$ , where  $T = (2\pi + \theta)/\beta$ , and after the impact the state position is  $r(\phi_0 + 2\pi^+) = (1 + k)r(\phi_0 + 2\pi) = (1 + k)e^{-\alpha T}r_0$ .

We construct the Poincaré map on the line  $\Gamma'$  and denote

$$q := \frac{r(\phi_0 + 2\pi^+)}{r(\phi_0 - \theta)} = (1 + k)e^{-\alpha T}, \tag{16}$$

from which we easily see that the following theorem holds.

**Theorem 2.1.** *If*

- (a)  $q = 1$ , then all solutions of (15) with the initial conditions on  $\Gamma'$  are  $T$ -periodic;
- (b)  $q < 1$ , then all solutions of (15) with the initial conditions on  $\Gamma'$  spiral in towards the origin;
- (c)  $q > 1$ , then all solutions of (15) with the initial conditions on  $\Gamma'$  move away from the origin.

**Remark 2.1.** In this study, for given  $\alpha, \beta$  and  $\theta$ , we fix the number  $I$  as  $I = (e^{\alpha T} - \cos \theta) / (\cos \theta - e^{-\alpha T}) - 1$  so that  $q = 1$  and hence part (a) of the Theorem 2.1 holds (see Fig. 2). Since  $q \neq 1$  is the non-critical case, when the phase portrait is persistent under perturbations, our present interest is only with the case (a) of the Theorem 2.1.

**Remark 2.2.** If  $q$ , which is defined in (16), is less than 1, then any solution of (14) with the initial condition on  $\Gamma'$  spirals in towards the origin, and if  $q > 1$  then the solutions move away from the origin. On the other hand, when  $q = 1$  we have the critical case. That is, a solution of (14) with the initial condition on  $\Gamma'$  may be periodic, it may spiral in towards the origin or it may move away from the origin. In this study, as mentioned before, the case  $q \neq 1$  is not of our interest and the critical case is investigated below.

## 2.4. Hopf bifurcation

In this section, we consider (13) again:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + \tilde{\mathcal{R}}(\rho, \varphi, \mu), \quad \varphi \neq \phi_0(\mu), \\ \Delta\rho|_{\varphi=\phi_0(\mu)} &= k(\mu)\rho + \tilde{W}(\rho, \mu). \end{aligned} \quad (17)$$

We consider the corresponding linearized system around the trivial solution:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho, \quad \varphi \neq \phi_0(\mu), \\ \Delta\rho|_{\varphi=\phi_0(\mu)} &= k(\mu)\rho. \end{aligned} \quad (18)$$

We construct the Poincaré map on the line  $\Gamma'_0(\mu)$  and denote

$$q(\mu) = (1 + k(\mu))e^{-\alpha(\mu)T(\mu)}, \quad (19)$$

in the same way as we defined (16).

**Theorem 2.2.** *Assume that  $q(0) = 1$ ,  $q'(0) \neq 0$ . Then for sufficiently small  $r_0$ , there exists a function  $\mu = \delta(r_0)$  with  $\delta(0) = 0$  such that the solution  $r(\phi, r_0, \delta(r_0))$  of (9) is periodic with a period  $T = (2\pi + \theta)/\beta + o(|\mu|)$ . Furthermore, if solutions of (14) spiral in towards the origin, then this periodic solution is a discontinuous limit cycle.*

**Proof.** Let  $\rho(\varphi, r_0, \mu)$  be a solution of (17). Because of the analyticity of solutions [3], we have

$$\rho(2\pi, r_0, \mu) = \sum_{i=1}^{\infty} a_i(\mu)r_0^i$$

where  $a_i(\mu) = \sum_{j=0}^{\infty} a_{ij}\mu^j$ ,  $a_{10} = q(0) = 1$ ,  $a_{11} = q'(0) \neq 0$ . Define

$$\begin{aligned} \mathcal{V}(r_0, \mu) &= \rho(2\pi, r_0, \mu) - r_0 \\ &= q'(0)\mu r_0 + \sum_{i=2}^{\infty} a_{i0}r_0^i + r_0\mu^2 M_1(r_0, \mu) + r_0^2\mu M_2(r_0, \mu) \end{aligned}$$

where  $M_1$  and  $M_2$  are analytic functions of  $r_0, \mu$  in a small neighborhood of the  $(0, 0)$ . When the bifurcation equation,  $\mathcal{V}(r_0, \mu) = 0$ , is simplified by  $r_0$ , one can write

$$\mathcal{H}(r_0, \mu) = 0 \quad (20)$$

where

$$\mathcal{H}(r_0, \mu) = q'(0)\mu + \sum_{i=2}^{\infty} a_{i0}r_0^{i-1} + \mu^2 M_1(r_0, \mu) + r_0\mu M_2(r_0, \mu).$$

By the implicit function theorem, since

$$\mathcal{H}(0, 0) = 0, \quad \frac{\partial \mathcal{H}(r_0, \mu)}{\partial \mu} = q'(0) \neq 0,$$

for sufficiently small  $r_0$  there exists a function  $\mu = \delta(r_0)$  with  $\delta(0) = 0$  such that  $r(\phi, r_0, \delta(r_0))$  is a periodic solution. If we assume that  $a_{i0} = 0$  for  $i = 2, \dots, \ell - 1$  and  $a_{\ell 0} \neq 0$ , then from (20) one can obtain that

$$\delta(r_0) = -\frac{a_{\ell 0}}{q'(0)}r_0^{\ell-1} + \sum_{i=\ell}^{\infty} \delta_i r_0^i. \quad (21)$$



Analyzing the last expression one can conclude that the bifurcation of periodic solution exists if a stable focus for  $\mu = 0$  is unstable for  $\mu \neq 0$  and vice versa. Let  $\rho(\varphi) = \rho(\varphi, \bar{r}_0, \bar{\mu})$  be a periodic solution of (17). It is known that the trajectory is limit cycle if

$$\frac{\partial \mathcal{V}(\bar{r}_0, \bar{\mu})}{\partial r_0} < 0. \tag{22}$$

Now,

$$\frac{\partial \mathcal{V}(r_0, \mu)}{\partial r_0} = q'(0)\mu + \sum_{i=2}^{\infty} i a_{i0} r_0^{i-1} + \mu^2 N_1(r_0, \mu) + r_0 \mu N_2(r_0, \mu).$$

If  $a_{\ell 0}$  is the first nonzero element among  $a_{i0}$  and  $a_{\ell 0} < 0$ , then using (21) one can obtain

$$\frac{\partial \mathcal{V}(\bar{r}_0, \bar{\mu})}{\partial r_0} = (\ell - 1)a_{\ell 0} \bar{r}_0^{\ell-1} + Q(\bar{r}_0),$$

where  $Q$  starts with a member whose order is not less than  $\ell$ . Hence, (22) is valid. From the  $\psi$ -substitution and  $B$ -equivalence of (9) and (12) one can conclude that the theorem is proved.

Since the change of variables  $x_1 = r \cos \phi$ ,  $x_2 = r \sin \phi$  and  $y = x_2$ ,  $y' = \beta x_1 - \alpha x_2$  are one-to-one for  $\beta \neq 0$ , we see that the following theorem is valid.

**Theorem 2.3.** *Assume that  $q(0) = 1$ ,  $q'(0) \neq 0$ . Then for sufficiently small initial condition  $y_0 := (y(0), y'(0))$  there exists a function  $\mu = \delta(y_0)$  with  $\delta(0) = 0$  such that the solution  $y(t, 0, y_0, \mu)$  of (6) is periodic with a period  $T = (2\pi + \theta)/\beta + o(|\mu|)$ . Furthermore, if solutions of (6) with the initial point on  $\Gamma'_0(0)$  spiral in towards the origin, then this periodic solution is a discontinuous limit cycle.*

**Example 2.1.** In the following example we shall consider the model which is studied in many papers and books [7,8,13,14,25,26,34,36]. Here, we insert the impulse condition and consider

$$\begin{aligned} y'' + \varepsilon_1 y' + y &= \varepsilon_2 y^2 y', & (y, y') \notin \Gamma, \\ \Delta y' |_{(y,y') \in \Gamma} &= dy', \end{aligned} \tag{23}$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are some nonzero real numbers,  $\Gamma$  is the discontinuity set which is defined, in  $yy'$ -plane, by  $y = 0$ ,  $y' > 0$ . The nonperturbed system is written as

$$\begin{aligned} y'' + 2\alpha y' + (\alpha^2 + \beta^2)y &= 0, & (y, y') \notin \Gamma, \\ \Delta y' |_{(y,y') \in \Gamma} &= dy'. \end{aligned} \tag{24}$$

where  $\alpha = \varepsilon_1/2$ ,  $\beta = \sqrt{1 - \alpha^2}$ ,  $d = e^{2\pi\alpha/\beta} - 1$ . Note that the general solution of the differential equation without impulse condition in (24) is given by

$$y(t) = e^{-\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t)), \tag{25}$$

where  $C_1$  and  $C_2$  are arbitrary real constants. Let  $(0, y'_0)$  be any point on the line  $\Gamma' = \Gamma$ . That is, assume that  $y'_0 > 0$ . Then  $y(0) = 0$ ,  $y'(0) = y'_0$  in (25) gives us  $C_1 = 0$ ,  $C_2 = y'_0/\beta$ . Thus, we obtain

$$y(t) = \frac{y'_0 e^{-\alpha t} \sin(\beta t)}{\beta}.$$

Now, the first impact action takes place at time  $t = T$  where  $T > 0$  and  $y(T) = 0$ , which means  $T = 2\pi/\beta$ . At that time, we have  $y'(T) = e^{-2\pi\alpha/\beta} y'_0$ , and after the impact we have  $y'(T^+) = (1 + d)y'(T) = y'_0$ . Therefore, all solutions starting on  $\Gamma'$  are  $T = 2\pi/\beta$  periodic. One such solution with  $y'_0 = 0.06$  is depicted in Fig. 3.

The fact that for (24) the origin is a center makes it suitable for application of our theoretical results. For this reason, let us perturb (23) and bring it to the notation of system (2). That is, let us consider the model

$$\begin{aligned} y'' + 2\alpha y' + (\alpha^2 + \beta^2)y &= F(y, y', \mu), & (y, y') \notin \Gamma(\mu), \\ \Delta y' |_{(y,y') \in \Gamma(\mu)} &= cy + dy' + J(y, y', \mu), \end{aligned} \tag{26}$$

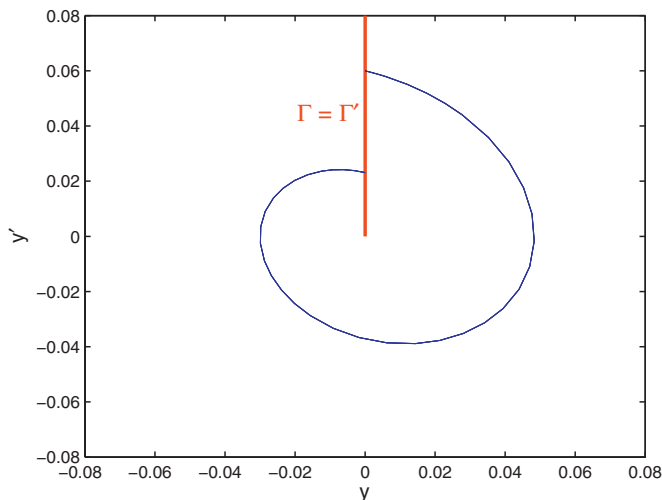


Fig. 3. A solution of the nonperturbed system (24) with the initial condition  $y(0)=0, y'(0)=0.06$ .

where  $\alpha=0.15, \beta = \sqrt{1 - \alpha^2}, F(y, y', \mu)=0.02\alpha y^2 y' - \mu y(2 + y + y'), \Gamma(\mu)$  is the curve  $\Gamma(\mu) = \{(y, y') \in \mathbb{R}^2 : y + 30\mu(y')^2 = 0, y' > 0\}, c = \alpha d, d = e^{2\pi\alpha/\beta} - 1$  and  $J(y, y', \mu) = -(2 + \mu)(y')^2$ . Note that (26) is a special case of (2) with  $m_1 = 1, m_2 = 0, \tau(y, y', \mu) = 30\mu(y')^2$ . The term  $cy$  in (26) has to be considered now as a small perturbation because of smallness of  $y$  as the first coordinate of points  $\Gamma(\mu)$ .

To prove the existence of a periodic solution for (26), we find that the generalized eigenvalue is

$$q(\mu) = \exp\left(2\pi\alpha\left(\frac{1}{\beta} - \frac{1}{\sqrt{\beta^2 + 2\mu}}\right)\right).$$

Then, one can easily find that  $q(0) = 1$  and  $q'(0) \neq 0$ . Therefore, by Theorem 2.3, for sufficiently small  $\mu$ , there exists a periodic solution with period  $\approx 40\pi/\sqrt{391}$ .

When  $\mu = 0$  in (26), the origin is a stable focus. This can be seen in simulations and the solution of (26) corresponding to  $\mu = 0$  with the initial condition  $y(0)=0, y'(0)=0.06$  is drawn in Fig. 4.

To see an application of Theorem 2.3, we take  $\mu = 0.08$ . By Theorem 2.3, we know that there exists a periodic solution corresponding to an initial value  $(y(0), y'(0))$  in a sufficiently small neighborhood of the origin. Two solutions

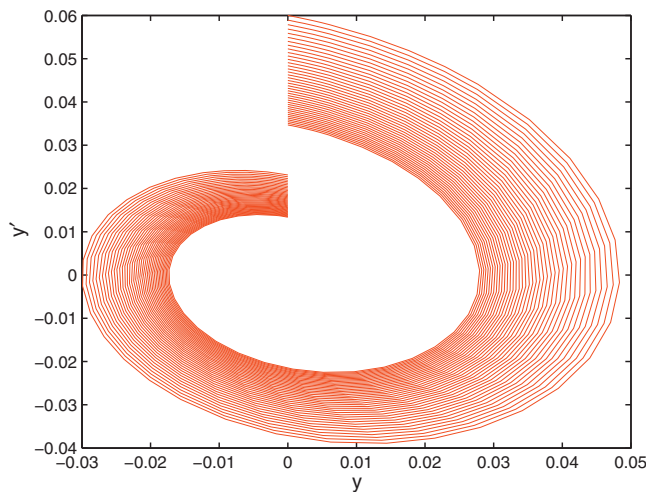


Fig. 4. A solution of the perturbed system with the initial condition  $y(0)=0, y'(0)=0.06$ .

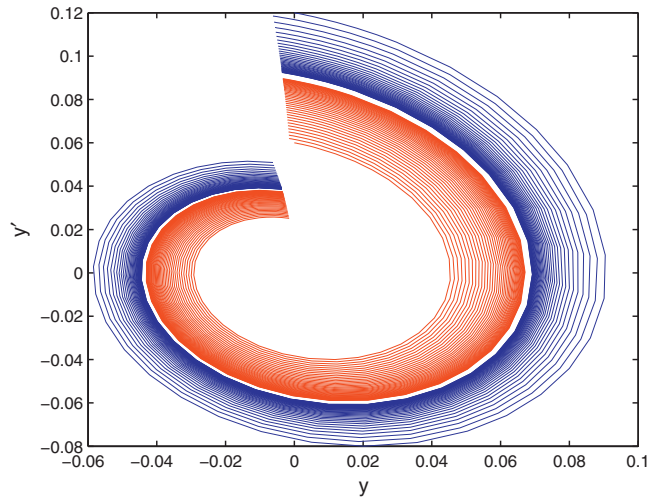


Fig. 5. An “inner” and an “outer” solution of (26). The inner solution is shown in red and it corresponds to the initial condition  $y(0) = 0, y'(0) = 0.06$ . The outer solution is shown in blue and it corresponds to the initial condition  $y(0) = 0, y'(0) = 0.12$ .

of (26) are drawn in Fig. 5. An “inner” solution with the initial condition  $y(0) = 0, y'(0) = 0.06$  is drawn in red curve and an “outer” solution with the initial condition  $y(0) = 0, y'(0) = 0.12$  is drawn in blue curve. These two solutions approach to the limit cycle from inside and outside, respectively. Moreover, by Theorem 2.3, the period of the discontinuous limit cycle is approximately  $40\pi/\sqrt{391}$ .

Note that the differential equation of (26) for  $\mu = 0$  becomes (1) with  $\varepsilon = 0.3$ . This example shows the application importance of our results.

System (26) for  $\mu = 0$  is one of the widely investigated models of the mechanisms with impacts determined by the Newton’s law of restitution. These models have been studied in the books [13,34] and papers cited therein. One should mention that surfaces of discontinuity in these results are flat. However, in our paper, it is the first time, we consider the surface of discontinuity as perturbed nonlinearly.

### 3. Center manifold

In this section, we begin our development of the techniques and extend our results to coupled oscillators. We show the existence of a center manifold. Consider

$$\begin{aligned}
 y'' + 2\alpha y' + (\alpha^2 + \beta^2)y &= F_1(y, y', z, z', \mu), \\
 z'' + 2\gamma z' + (\gamma^2 + \sigma^2)z &= F_2(y, y', z, z', \mu), \quad (y, y') \notin \Gamma(\mu), \\
 \Delta y'|_{(y,y') \in \Gamma(\mu)} &= cy + dy' + J(y, y', \mu),
 \end{aligned}
 \tag{27}$$

where  $y, y'$  are the components of one oscillator, say (A), and  $z, z'$  are the components of another oscillator, say (B).

By using the means of the center manifold theorem [32] and its role for the Hopf bifurcation in multidimensional systems, one can predict that the system of oscillators (A) and (B) admits a limit cycle. Indeed, looking at the results of our simulations given in Figs. 6–8, one can see that for the oscillator (A) we have a discontinuous limit cycle, as in Example 2.1, and for the oscillator (B) we have a continuous limit cycle. That is, for the whole system the periodic trajectory  $(y, z)$  is a discontinuous limit cycle.

As we did to obtain (6), we write (27) in the form

$$\begin{aligned}
 y'' + 2\alpha(\mu)y' + (\alpha^2(\mu) + \beta^2(\mu))y &= G_1(y, y', z, z', \mu), \\
 z'' + 2\gamma(\mu)z' + (\gamma^2(\mu) + \sigma^2(\mu))z &= G_2(y, y', z, z', \mu), \quad (y, y') \notin \Gamma(\mu), \\
 \Delta y'|_{(y,y') \in \Gamma(\mu)} &= cy + dy' + J(y, y', \mu).
 \end{aligned}
 \tag{28}$$

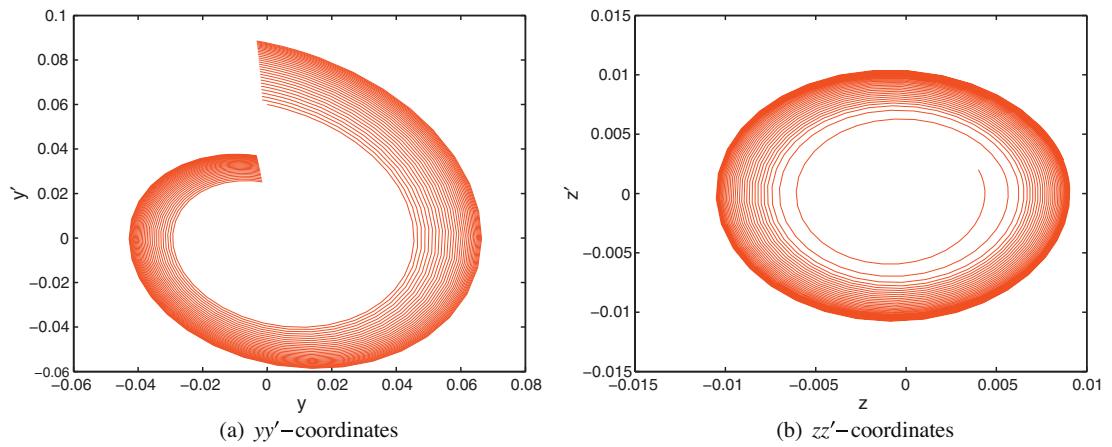


Fig. 6. An “inner” solution of (37) for  $\mu = 0.08$  corresponding to the initial condition  $y(0) = 0, y'(0) = 0.06, z(0) = 0.004, z'(0) = 0.002$ .

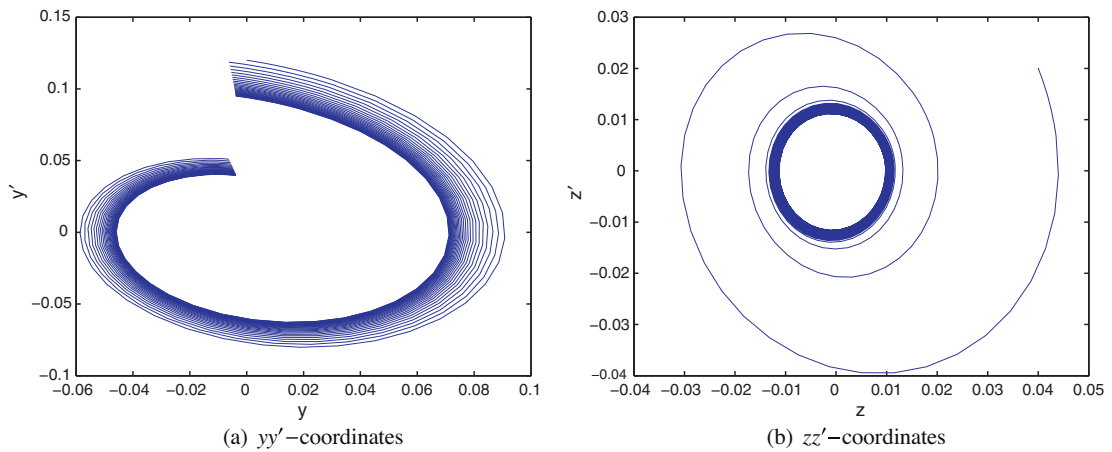


Fig. 7. An “outer” solution of (37) for  $\mu = 0.08$  corresponding to the initial condition  $y(0) = 0, y'(0) = 0.12, z(0) = 0.04, z'(0) = 0.02$ .

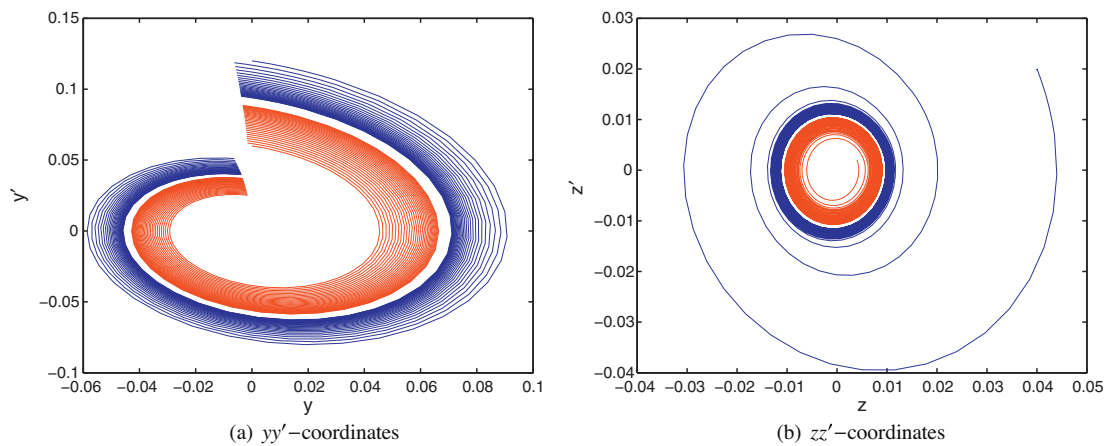


Fig. 8. The “inner” and “outer” solutions of (37) given in Figs. 6 and 7 are shown in the same picture.

Let  $y_1 = (\alpha(\mu)y + y')/\beta(\mu)$ ,  $y_2 = y$ ,  $z_1 = (\gamma(\mu)z + z')/\sigma(\mu)$ ,  $z_2 = z$ . Then (28) becomes

$$\begin{aligned} y_1' &= -\alpha(\mu)y_1 - \beta(\mu)y_2 + H_1(y_1, y_2, z_1, z_2, \mu), \\ y_2' &= \beta(\mu)y_1 - \alpha(\mu)y_2, \\ z_1' &= -\gamma(\mu)z_1 - \sigma(\mu)z_2 + H_2(y_1, y_2, z_1, z_2, \mu), \\ z_2' &= \sigma(\mu)z_1 - \gamma(\mu)z_2, \quad (y_1, y_2) \notin \Gamma(\mu), \\ \Delta y_1|_{(y_1, y_2) \in \Gamma(\mu)} &= Iy_1 + K(y_1, y_2, \mu). \end{aligned} \tag{29}$$

We now use the cylindrical coordinates. That is, we use the polar coordinates for the oscillator (A). Then we eliminate the variable  $t$ , and obtain

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda(\mu)r + \mathcal{R}(r, \phi, z_1, z_2, \mu), \\ \frac{dz_1}{d\phi} &= -\tilde{\gamma}(\mu)z_1 - \tilde{\sigma}(\mu)z_2 + \mathcal{R}_2(r, \phi, z_1, z_2, \mu), \\ \frac{dz_2}{d\phi} &= \tilde{\sigma}(\mu)z_1 - \tilde{\gamma}(\mu)z_2, \quad \phi \neq \phi_0(\mu) + \xi_{2j}(r, \phi, \mu), \\ \Delta r|_{\phi=\phi_0(\mu)+\xi_{2j}(r, \phi, \mu)} &= k(\mu)r + R(r, \phi, \mu), \\ \Delta \phi|_{\phi=\phi_0(\mu)+\xi_{2j}(r, \phi, \mu)} &= -\theta(\mu)r + \Theta(r, \phi, \mu), \end{aligned} \tag{30}$$

where  $\tilde{\gamma}(\mu) = \gamma(\mu)/\beta(\mu)$ ,  $\tilde{\sigma}(\mu) = \sigma(\mu)/\beta(\mu)$ , and all other elements except for  $\mathcal{R}_2$  are the same as in (9). Using B-equivalence and  $\psi$ -substitution, we get the following system:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + \tilde{\mathcal{R}}(\rho, \varphi, z_1, z_2, \mu), \\ \frac{dz_1}{d\varphi} &= -\tilde{\gamma}(\mu)z_1 - \tilde{\sigma}(\mu)z_2 + \tilde{\mathcal{R}}_2(\rho, \varphi, z_1, z_2, \mu), \\ \frac{dz_2}{d\varphi} &= \tilde{\sigma}(\mu)z_1 - \tilde{\gamma}(\mu)z_2, \quad \varphi \neq \phi_0(\mu), \\ \Delta \rho|_{\varphi=\phi_0(\mu)} &= k(\mu)\rho + \tilde{W}(\rho, \mu). \end{aligned} \tag{31}$$

The corresponding linearized system around the origin is

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho, \\ \frac{dz_1}{d\varphi} &= -\tilde{\gamma}(\mu)z_1 - \tilde{\sigma}(\mu)z_2, \\ \frac{dz_2}{d\varphi} &= \tilde{\sigma}(\mu)z_1 - \tilde{\gamma}(\mu)z_2, \quad \varphi \neq \phi_0(\mu), \\ \Delta \rho|_{\varphi=\phi_0(\mu)} &= k(\mu)\rho. \end{aligned} \tag{32}$$

Like we obtained (19), we define

$$\begin{aligned} q_1(\mu) &= (1 + k(\mu))e^{-\alpha(\mu)T(\mu)}, \\ q_2(\mu) &= e^{-\tilde{\gamma}(\mu)T(\mu)}. \end{aligned} \tag{33}$$

Note that  $q_1(0) = 1$  and hence we have the critical case for the first oscillator and  $q_2(0) < 1$  if and only if  $\tilde{\gamma}(0) > 0$ . For our system, we assume that  $q_1(0) = 1$  and  $q_2(0) < 1$ .

By using the formulas for the integral manifolds developed in [32], one can see that system (31) has a center manifold  $S_0(\mu) := \{(\rho, \varphi, u) : u = \Phi_0(\varphi, \rho, \mu)\}$  and a stable manifold  $S_-(\mu) := \{(\rho, \varphi, u) : \rho = \Phi_-(\varphi, u, \mu)\}$  where

$$\Phi_0(\varphi, \rho, \mu) = \int_{-\infty}^{\varphi} \pi_0(\varphi, s, \mu) \tilde{\mathcal{R}}_2(\rho(s, \varphi, \rho, \mu), s, u(s, \varphi, \rho, \mu), \mu) ds \tag{34}$$

and

$$\Phi_-(\varphi, u, \mu) = - \int_{\varphi}^{\infty} \pi_-(\varphi, s, \mu) \tilde{\mathcal{R}}_1(\rho(s, \varphi, u, \mu), s, u(s, \varphi, u, \mu), \mu) ds + \sum_{\varphi_i < \varphi} \pi_-(\varphi, \varphi_i^+, \mu) \tilde{W}(\rho(s, \varphi, u, \mu), \mu) \tag{35}$$

in which  $\varphi_i = \varphi + 2\pi i, u = (z_1, z_2)$ ,

$$\pi_0(\varphi, s, \mu) = e^{-\tilde{\gamma}(\mu)(\varphi-s)} \begin{bmatrix} \cos(\tilde{\sigma}(\mu) - s) & -\sin(\tilde{\sigma}(\mu) - s) \\ \sin(\tilde{\sigma}(\mu) - s) & \cos(\tilde{\sigma}(\mu) - s) \end{bmatrix}$$

and

$$\pi_-(\varphi, s, \mu) = e^{\lambda(\mu)(\varphi-s)} \prod_{s \leq \varphi_j(\mu) < \varphi} (1 + k(\mu)).$$

In (34), the pair  $(\rho(s, \varphi, \rho, \mu), u(s, \varphi, \rho, \mu))$  denotes a solution of (31) with  $\rho(s, \varphi, \rho, \mu) = \rho$ . Similarly, in (35), the pair  $(\rho(s, \varphi, u, \mu), u(s, \varphi, u, \mu))$  denotes a solution of (31) such that  $u(s, \varphi, u, \mu) = u$ .

On the local center manifold,  $S_0(\mu)$ , the first coordinate of the solutions of (31) satisfies the following system

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + \tilde{\mathcal{R}}(\rho, \varphi, \Phi_0(\varphi, \rho, \mu), \mu), \quad \varphi \neq \varphi_0(\mu), \\ \Delta\rho|_{\varphi=\varphi_0(\mu)} &= k(\mu)\rho + \tilde{W}(\rho, \mu). \end{aligned} \tag{36}$$

By Theorem 2.3, we know that, for sufficiently small  $\mu$ , system (36) has a periodic solution with period  $T = (2\pi + \theta)/\beta + o(\mu)$ . That is, on the local center manifold  $S_0(\mu)$ , the  $\rho$ -coordinate of a solution of (31) is  $T$ -periodic. Because of the  $T$ -periodic properties of the right side functions, one can show that  $z_1$  and  $z_2$  components of a solution of (31) are also  $T$ -periodic when the  $\rho$ -coordinate is. Thus, we have the following theorem.

**Theorem 3.1.** *Assume that  $q_1(0) = 1, q_1'(0) \neq 0, q_2(0) < 1$ . Then for sufficiently small initial condition  $(y_0, z_0) := (y(0), y'(0), z(0), z'(0))$  there exists a function  $\mu = \delta(y_0, z_0)$  such that the solution  $(y(t, 0, y_0, \mu), z(t, 0, z_0, \mu))$  of (27) is periodic with a period  $T = (2\pi + \theta)/\beta + o(|\mu|)$ . Furthermore, if solutions of (27) with the  $y_0$ -component of the initial point on  $\Gamma'(0)$  spiral in towards the origin, then this periodic solution is a discontinuous limit cycle.*

**Example 3.1.** Let us develop the model studied in Example 2.1 further to two coupled oscillators where one of the oscillators is subdued to the impacts. For this reason, consider

$$\begin{aligned} y'' + 2\alpha y' + (\alpha^2 + \beta^2)y &= F_1(y, y', z, z', \mu), \\ z'' + 2\gamma z' + (\gamma^2 + \sigma^2)z &= F_2(y, y', z, z', \mu), \quad (y, y') \notin \Gamma(\mu), \\ \Delta y'|_{(y,y') \in \Gamma(\mu)} &= dy' + J(y, y', \mu), \end{aligned} \tag{37}$$

where  $\gamma = 0.2, \sigma = 1$ ,

$$\begin{aligned} F_1(y, y', z, z', \mu) &= 0.02\alpha y^2 y' - \mu y(2 + y + z + (z')^2), \\ F_2(y, y', z, z', \mu) &= \gamma y z' - \mu y(1 + z^2 + (z')^2), \end{aligned}$$

and all other elements are the same as in Example 2.1.

As evaluated before, we have  $q_1(0) = 1, q_1'(0) \neq 0$ , and  $\gamma > 0$  implies that  $q_2(0) < 1$ . Thus, by Theorem 3.1, for sufficiently small  $\mu$ , there exists a periodic solution with period  $\approx 40\pi/\sqrt{391}$ . Let  $\mu = 0.08$ . An “inner” solution with the initial condition  $y(0) = 0, y'(0) = 0.06, z(0) = 0.004, z'(0) = 0.002$  is represented in Fig. 6 and an “outer” solution

with the initial condition  $y(0) = 0$ ,  $y'(0) = 0.12$ ,  $z(0) = 0.04$ ,  $z'(0) = 0.02$  is shown in Fig. 7. These two solutions approach to the limit cycle from inside and outside, respectively.

#### 4. Conclusion

Many evolutionary processes are subject to short-term perturbation whose duration is negligible when compared to that of the whole process. This perturbation results in a change in the state of the process. This change can be at fixed moments or when the state process meets a certain set of discontinuity. These systems model a variety of problems in mechanics, electronics, physics, chemistry, medicine, etc. [7–11,13,15,17,18,21,23,24,29,33,35,37].

In most of the references cited here, the impulse action or the change in the phase space takes place on a flat surface. The theory in which nonlinear surfaces of discontinuity are present has not been investigated fully because of the lack of theoretical results. The problem of discontinuous models where the surfaces of discontinuities are not flat is very actual because of natural possibilities of perturbations. It is natural that one should involve perturbation not only into the differential equation or in the impulse function, but also into the equations for the surfaces of discontinuity.

The discontinuities of the equation which determines the moments of jumps are investigated in many papers and books [34]. In most general form, the results are formulated and expressed in [3], and the present results widely use this information.

In this paper, the method of  $B$ -equivalence and  $\psi$ -substitution [1–4,6] are used effectively to observe the Hopf bifurcation of periodic solution. We proved the existence of discontinuous limit cycle for the Van der Pol equation performing impacts on surfaces. We extended the results to two coupled oscillators through the application of the center manifold theory [32]. We plan to extend these theoretical results to arbitrary dimension and apply them to well known discontinuous mechanical models.

One should mention that we consider the Hopf bifurcation without reduction to the problem of Hopf bifurcation of the maps as it is usually done in the literature. So, the system which admits the origin as a center in our case is nonlinear one (15), despite its elements are linear. This approach to the investigation is respectively new and promissive.

Based on the present results one can investigate multi oscillatory system where not only one of the oscillators is discontinuous but several of them are discontinuous [8,16,19,31].

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#### References

- [1] M.U. Akhmet, Nonlinear dynamics with piecewise constant argument, submitted for publication.
- [2] M.U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, *Nonlinear Analysis* 60 (2005) 163–178.
- [3] M.U. Akhmet, *Principles of Discontinuous Dynamical Systems*, Springer, New York, 2010.
- [4] M.U. Akhmet, M. Turan, The differential equations on time scales through impulsive differential equations, *Nonlinear Analysis* 65 (2006) 2043–2060.
- [5] M.U. Akhmet, M. Turan, Bifurcation of three-dimensional discontinuous cycles, *Nonlinear Analysis: Theory, Methods and Applications* 71 (2009) e2090–e2102.
- [6] M.U. Akhmet, M. Turan, Differential equations on variable time scales, *Nonlinear Analysis: Theory, Methods and Applications* 70 (2009) 1175–1192.
- [7] A.A. Andronov, C.E. Chaikin, *Theory of Oscillations*, Princeton Uni. Press, Princeton, 1949.
- [8] J. Awrejcewicz, C.H. Lamarque, *Bifurcation and Chaos in Nonsmooth Mechanical Systems*, World Scientific, Singapore, 2003.
- [9] S. Banerjee, J. Ing, E. Pavlovskaja, M. Wiercigroch, R.K. Reddy, Invisible grazings and dangerous bifurcations in impacting systems: the problem of narrow-band chaos, *Physical Review E* 79 (2009) 037201, URL <http://link.aps.org/doi/10.1103/PhysRevE.79.037201>.
- [10] V. Carmona, E. Freire, E. Ponce, Limit cycle bifurcation in 3d continuous piecewise linear systems with two zones. Application to Chua's circuit, *International Journal of Bifurcation and Chaos* 15 (10) (2005) 3153–3164.
- [11] C.-Y. Cheng, Induction of Hopf bifurcation and oscillation death by delays in coupled networks, *Physics Letters A* 374 (2009) 178–185.
- [12] W. Chin, E. Ott, H.E. Nusse, C. Grebogi, Grazing bifurcations in impact oscillators, *Physical Review E* 50 (3) (1994) 4427–4444.
- [13] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piece-wise Smooth Dynamical Systems*, Springer, London, 2008.



- [14] M. Feckan, Dynamical systems with discontinuities, *Dynamics of Continuous Discrete and Impulsive Systems Series A: Mathematical Analysis* 16 (2009) 789–809.
- [15] M.I. Feigin, Doubling dynamics of the oscillation period with C-bifurcation in piecewise-continuous system, *Applied Mathematics and Mechanics* 34 (1970) 822–829.
- [16] R.J. Field, E. Koros, R.M. Noyes, Oscillations in chemical systems. II. Through analysis of temporal oscillations in the bromate–cerium–malonic acid system, *Journal of the American Chemical Society* 94 (1972) 8649.
- [17] M.H. Fredriksson, A.B. Nordmark, Bifurcations caused by grazing incidence in many degree of freedom impact oscillators, *Proceedings of the Royal Society A* 453 (1997) 1261–1275.
- [18] E. Freire, The focus-center-limit cycle bifurcation in symmetric 3d piecewise linear systems, *SIAM Journal of Applied Mathematics* 65 (6) (2005) 1933–1951.
- [19] H. Fujii, T. Yamada, Phase difference locking of coupled oscillating chemical systems, *Journal of Chemical Physics* 69 (1978) 3830.
- [20] C. Glocker, F. Pfeiffer, Multiple impacts with friction in rigid multibody systems, *Nonlinear Dynamics* 7 (4) (1995) 471–497.
- [21] F.C. Hoppensteadt, E.M. Izhikevich, *Weakly Connected Neural Networks*, Springer, New York, 1997.
- [22] A.P. Ivanov, Impact oscillations: Linear theory of stability and bifurcations, *Journal of Sound and Vibration* 178 (3) (1994) 361–378.
- [23] S.G. Kryzhevich, Grazing bifurcation and chaotic oscillations of vibro-impact systems with one degree of freedom, *Journal of Applied Mathematics and Mechanics* 72 (4) (2008) 383–390.
- [24] T. Kupper, S. Moritz, Generalized Hopf bifurcation for non-smooth planar systems, *Philosophical Transactions of the Royal Society of London* 359 (2001) 2483–2496.
- [25] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer-Verlag, New York, 1995.
- [26] R.I. Leine, D.H.V. Campen, B.L.V. de Vrande, Bifurcations in nonlinear discontinuous systems, *Nonlinear Dynamics* 23 (2000) 105–164.
- [27] R.I. Leine, N.V. de Wouw, Stability properties of equilibrium sets of non-linear mechanical systems with dry friction and impact, *Nonlinear Dynamics* 51 (4) (2008) 551–583.
- [28] J.E. Marsden, M. McCracken, *The Hopf Bifurcation and Its Applications*, Springer, New York, 1976.
- [29] A.H. Nayfeh, B. Balachandran, *Applied Nonlinear Dynamics*, John Wiley & Sons, Inc, New York, 1995.
- [30] A.B. Nordmark, Non-periodic motion caused by grazing incidence in an impact oscillator, *Journal of Sound and Vibration* 145 (1991) 279–297.
- [31] F. Pfeiffer, C. Glocker, *Multibody Dynamics with Unilateral Contacts*, John Wiley & Sons, New York, 1996.
- [32] V.A. Pliss, *Integral Sets of Periodic Systems of Differential Equations*, Izdat, Nauka, Moscow, 1977.
- [33] T. Rui-Lan, C. Qing-Jie, L. Zhi-Xin, Hopf bifurcations for the recently proposed smooth-and-discontinuous oscillator, *Chinese Physics Letters* 27 (7) (2010) 074701.
- [34] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.
- [35] C. Sun, M. Han, X. Pang, Global Hopf bifurcation analysis on a BAM neural network with delays, *Physics Letters A* 360 (2007) 689–695.
- [36] J.M.T. Thompson, H.B. Stewart, *Nonlinear Dynamics and Chaos*, John Wiley, England, 2002.
- [37] G.S. Whiston, Global dynamics of a vibro-impacting linear oscillator, *Journal of Sound and Vibration* 118 (3) (1987) 395–429.