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Letters

Generation of cyclic/toroidal chaos by Hopfield neural networks

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ABSTRACT

We discuss the appearance of cyclic and toroidal chaos in Hopfield neural networks. The theoretical results may strongly relate to investigations of brain activities performed by neurobiologists. As new phenomena, extension of chaos by entrainment of several limit cycles as well as the attraction of cyclic chaos by an equilibrium are discussed. Appropriate simulations that support the theoretical results are depicted. Stabilization of tori in a chaotic attractor is realized not only for neural networks, but also for differential equations theory, and this phenomenon has never been reported before in the literature. It is demonstrated that the proposed chaos generation technique cannot be considered as generalized synchronization.

1. Introduction

There is a certain adequacy for the real world and its reflection by brain activities. The presence of chaos in neural networks is useful for separating image segments [1], information processing [2,3] and synchronization [4–7]. Either the chaos can be generated by a neural network itself (endogenous chaos), or a chaotic influence outside of the neural network can be realized in its output (exogenous chaos). The endogenous chaos in neural networks has been widely investigated in the literature [8–24], but the latter has not been effectively discussed yet. This is not because the problem is not natural, but the absence of a rigorously developed input/output mechanism for the phenomenon seems to be the reason. This is why we were attracted by the problem of chaos generation.

In their experiments, Skarda and Freeman [25] obtained different kinds of electroencephalogram (EEG) signals when known and unknown odorants were given to a rabbit. For known odorants, the signals were in the form of a *limit cycle*, but for unknown ones, they were *chaotic*. According to the experimental results, it was proposed that deterministic chaos is utilized in neural activities for learning new sensory patterns as well as ensuring continual access to previously learned sensory patterns. The roles of chaos for brain behavior have been investigated in many papers. For example, Watanabe et al. [26] demonstrated that the chaotic dynamics works as means to learn new patterns and increases the *memory capacity* of neural networks. The group of theorists, Guevara et al. [27], suggested that chaotic behavior may be responsible for dynamical diseases such as schizophrenia, insomnia, epilepsy and dyskinesia. It was shown in the paper [27] that the periodic forcing of neural oscillator models can lead to chaos. This is similar to the case that was primarily observed in electrical devices through Van der Pol and Duffing oscillators in pioneer papers [28–32]. Actually, this is not surprising since brain activities can be mostly considered as electrical processes.

A sensory cortex is conceived in [33] as a global attractor with many "wings". When the cortex is at rest, the wings are shut. When a known stimulus arrives, the system moves to an appropriate wing and a burst of oscillation is observed. In paper [34], it was revealed that "each of the wings are either a near-limit cycle (a narrow band chaos) or a broad band chaos." One should emphasize that the near-limit cycle chaos can result from the entrainment of limit cycles by chaos, which is theoretically proved in [35] and considered as one of the main ways of chaos generation in the present research.

Discussing wings as neural networks, one can suppose that there is the opportunity of chaos production by a wing itself and extension of chaos from one wing to another. The latter case has not been considered in the literature yet, at least mathematically. That is why we decided to investigate the problem in the present study. More precisely, for the first time in the literature, we consider the extension of chaos in the following ways: (i) entrainment of a limit cycle by chaos, (ii) attraction of a cyclic chaos by an equilibrium, (iii) entrainment of two and more limit cycles by chaos, and (iv) attractions of chaotic cycles by an

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equilibrium. These chaos extension types provide us with the mathematical support for Freeman's "wings" of a sensory cortex. The extension of chaos has not been considered in the previous investigations, and we hope that the rigorous mathematical background of the phenomenon may give a positive effect for the researches of neurobiologists. This is also true for our results concerning quasi-periodic and almost periodic motions in the basis of chaotic attractors. Since the appearances of limit cycle and near-limit cycle chaos were experimentally observed in the studies [25,34], one can hypothesize that both limit tori and near-torus chaos can be dynamical representatives of brain processes.

The present study is suggested as an application of our investigations about chaos extension developed in the papers [35–37] to give an additional mathematical light on the ideas developed by neurobiologists, primarily, Freeman and his collaborators [11,33,34,38–40]. The main dynamical result considered in the present study is the entrainment by chaos, which is understood as *the deformation of limit cycles to chaotic cycles*. Our results are useful for analyzing chaos extension among collectives of neural networks based on generation of chaos by input/output mechanisms built through differential equations. According to Skarda and Freeman [25], limit cycles and chaotic dynamics are of prime importance in odor recognition. Moreover, it was observed in [25] that the brain's EEG activity changes from limit cyclical to near-cycle chaotic if a familiar odor was replaced by an unknown one. This can be interpreted through our paper [35] as the chaotification of limit cycles. Additionally, if we accept that complexity of chaos is important for the memory capacity, then one can suppose that to increase a memory we need to do the same with the complexity of chaos. From this point of view, it is interesting to say about regular unstable motions which constitute a basis (skeleton) of chaotic attractors. These are usually assumed to be periodic motions [41–43]. Beside the periodic motions, quasi-periodic, almost periodic and recurrent motions can also be considered as a basis of chaos [44–46]. As chaos increases the capacity of memorizing [25,26], one can suppose that chaos with the basis of quasi-periodic motions provides a memory with a larger capacity than that with periodic motions. This is true if we compare chaos with quasi-periodic unstable motions with one having a skeleton of almost periodic motions. That is why the problem of chaos generation by neural networks which is based on unstable quasi-periodic or almost periodic solutions is of strong importance. In our paper, it is shown that one can create quasi-periodic motions in chaotic attractors as well as join different quasi-periodic motions to obtain quasi-periodic motions with a larger number of incommensurate frequencies. Moreover, we discuss the problem of chaos control, which can also be considered as theoretical basis of learning and recognition, if one accepts the ideas in the papers [25,34,38–40]. We suggest that the appearance of limit cycles in experiments with brain behavior [25,26,34] results from the stabilization of one of the unstable periodic solutions, which are already present in a wing. This stabilization can be done either by external perturbation or by control (of Pyragas type [47]), which is triggered by stimuli.

To have a unity and uniform delivering in the paper, the discussions are developed by using Hopfield neural networks (HNNs) [48–51], but they can also be realized for other types of neural networks [52–59].

HNNs [48–51] are continuous-time dynamical systems described by the following nonlinear ordinary differential equations:

$$C_i\frac{dp_i}{dt} = -\frac{p_i}{R_i} + \sum_{j=1}^{N} w_{ij} f_j(p_j) + I_i, \quad i = 1,2,\ldots,N, \tag{1.1}$$

where N is the number of neurons, p_i is the total input to neuron i, the bounded monotonic differentiable function f_j is the activation function acted on neuron j, C_i and R_i are the parameters corresponding to a capacitance and a resistance, I_i is the external input of neuron i and w_{ij} is the synaptic connection value between neuron i and neuron j.

In an equivalent form, the HNN (1.1) can be represented as

$$\dot{p} = -Cp + Wf(p) + I,$$

where $p = (p_1, p_2, \ldots, p_N)^T$, the diagonal matrix $C = diag\{c_1, c_2, \ldots, c_N\}$, which is associated with C_i and R_i, has positive diagonal entries, $f(p) = (f_1(p_1), f_2(p_2), \ldots, f_N(p_N))^T$, $W = (w_{ij})_{N\times N}$ is the connection matrix and $I = (I_1, I_2, \ldots, I_N)^T$ is the external input vector.

Weak synaptic connections between neurons are observable in the dynamics of brain, and a method to characterize the weakness of synaptic connections is to consider amplitudes of postsynaptic potentials measured in the soma of neurons while the neuron membrane potential is far below the threshold value [60]. The brain units such as neurons, cortical columns and neuronal modules are supposed to be weakly connected and modeled as autonomous quasi-periodic oscillators in the paper [61]. McNaughton et al. [62] revealed weak synaptic connections in the hippocampal cells by means of the investigation of excitatory postsynaptic potentials. Moreover, weak interactions between neurons in the cortex are observed by Abeles [63] as a result of the analysis of cross correlograms obtained from pairs of neurons. On the other hand, according to Pasemann et al. [64], periodic and quasi-periodic solutions in biological and artificial systems are of fundamental importance as they are associated with central pattern generators. Therefore, the investigations of coupled neural networks that possess periodic or quasi-periodic solutions with weak connections are of prime importance.

In the present study, we establish weak connections between two HNNs, one with a chaotic attractor and the other one with an attracting limit cycle or attracting torus. As a result we obtain a chaotic cycle/torus, that is, motions that behave chaotically around the limit cycle or torus.

Stability is one of the main properties which are suggested to be started with pioneer papers [48]. It attracts the attention of other authors nowadays [65–72]. However, starting with results on chaos, the role of papers on unstable motions and their stabilization has been increased significantly.

In the literature, the generation of chaos has been considered within the scope of synchronization theory [73–79]. For two coupled systems to be synchronized, the chaos of the response system has to be asymptotically close to that of the driver. We do not use this proximity in our results, and we demonstrate that chaos generation around limit cycles and tori are not reducible to synchronization, in general.

2. Entrainment of limit cycles and tori by chaos in HNNs

Let us consider the HNN

$$\dot{x} = -Cx + Wf(x) + I, \tag{2.1}$$

where $x \in \mathbb{R}^m$, $C = diag\{c_1, c_2, \ldots, c_m\}$, $c_i > 0$ for each $i = 1,2,\ldots,m$, $W = (w_{ij})_{m\times m}$ is the connection matrix and I is the external input vector.

Next, we take into account the HNN

$$\dot{y} = -Dy + \overline{W}g(y) + \varepsilon h(x(t)), \tag{2.2}$$

where $y \in \mathbb{R}^n$, $x(t)$ are solutions of (2.1), ε is a nonzero constant, $h : \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function, $D = diag\{d_1, d_2, \ldots, d_n\}$, $d_i > 0$ for each $i = 1,2,\ldots,n$ and $\overline{W} = (\overline{w}_{ij})_{n\times n}$ is the connection matrix. It is worth noting that the unidirectionally coupled networks (2.1) and (2.2) have a skew product structure.

We mainly assume that the HNN

$$\dot{u} = -Du + \overline{W}g(u) \tag{2.3}$$

possesses an orbitally stable limit cycle.

On the other hand, we also assume that the network (2.1) admits a chaotic attractor, let us say a set in $\mathbb{R}^m$. Fix x_0 from the attractor and take a solution $x(t)$ of (2.1) with $x(0) = x_0$. Since we use the solution $x(t)$ as an external input in the network (2.2), we call it as *chaotic function*. The chaotic functions may be irregular as well as regular (periodic and unstable) [42,43,80,81].

The network (2.1) is called sensitive if there exist positive numbers ϵ_0 and Δ such that for an arbitrary positive number δ_0 and for each chaotic solution $x(t)$ of (2.1), there exist a chaotic solution $\overline{x}(t)$ of the same network and an interval $J \subset [0, \infty)$, with a length no less than Δ, such that $\|x(0) - \overline{x}(0)\| < \delta_0$ and $\|x(t) - \overline{x}(t)\| > \epsilon_0$ for all $t \in J$.

For a given chaotic solution $x(t)$ of (2.1), let us denote by $\phi_{x(t)}(t, y_0)$, $y_0 \in \mathbb{R}^n$, the solution of (2.2) with $\phi_{x(t)}(0, y_0) = y_0$. The network (2.2) replicates the sensitivity of (2.1) if there exist positive numbers ϵ_1 and $\overline{\Delta}$ such that for an arbitrary positive number δ_1 and for each solution $\phi_{x(t)}(t, y_0)$, there exist an interval $J^1 \subset [0, \infty)$, with a length no less than $\overline{\Delta}$, and a solution $\phi_{\overline{x}(t)}(t, y_1)$ such that $\|y_0 - y_1\| < \delta_1$ and $\|\phi_{x(t)}(t, y_0) - \phi_{\overline{x}(t)}(t, y_1)\| > \epsilon_1$ for all $t \in J^1$. Moreover, we say that the network (2.2) is chaotic if it replicates the sensitivity of (2.1) and the coupled system (2.1)+(2.2) possesses infinitely many unstable periodic solutions in a bounded region.

The following theorem is based on the entrainment of limit cycles by chaos considered in the paper [35], where the replication of sensitivity and the existence of infinitely many unstable periodic solutions were rigorously proved.

Theorem 2.1. *If there exists a positive number L such that $\|h(s_1) - h(s_2)\| \geq L\|s_1 - s_2\|$ for all $s_1, s_2 \in \mathbb{R}^m$ and the number $|\varepsilon|$ is sufficiently small, then there exists a neighborhood $\mathcal{N}$ of the orbitally stable limit cycle of (2.3) such that solutions of (2.2) which start inside $\mathcal{N}$ behave chaotically around the limit cycle. That is, the solutions are sensitive and there are infinitely many unstable periodic solutions.*

To illustrate the result of Theorem 2.1, let us consider the HNN [22]

$$\begin{aligned}
\dot{u}_1 &= -u_1 + 3.4\tanh(u_1) - 1.6\tanh(u_2) + 0.7\tanh(u_3)\\
\dot{u}_2 &= -u_2 + 2.5\tanh(u_1) + 0.95\tanh(u_3)\\
\dot{u}_3 &= -u_3 - 3.5\tanh(u_1) + 0.5\tanh(u_2),
\end{aligned} \tag{2.4}$$

which is in the form of (2.3). It is mentioned in [22] that the network (2.4) possesses a limit cycle with the Lyapunov exponents 0, −0.1356 and −0.1466. Therefore, 1 is a simple characteristic multiplier of the corresponding variational system, and the remaining characteristic multipliers are in modulus less than 1.

According to the Andronov–Witt Theorem [82], the limit cycle of (2.4) is orbitally stable.

Next, we take into account the following HNN:

$$\begin{aligned}
\dot{x}_1 &= -x_1 + 2\tanh(x_1) - 1.2\tanh(x_2)\\
\dot{x}_2 &= -x_2 + 2\tanh(x_1) + 1.71\tanh(x_2) + 1.15\tanh(x_3)\\
\dot{x}_3 &= -x_3 - 4.75\tanh(x_1) + 1.1\tanh(x_3).
\end{aligned} \tag{2.5}$$

In the paper [21], it is shown that the network (2.5) admits a positive Lyapunov exponent and possesses chaotic motions. We will use it as a system of the form (2.1), which entrains the limit cycle of (2.4) by the chaos.

Making use of the solutions of (2.5) as external inputs for (2.4), we set up the following HNN:

$$\begin{aligned}
\dot{y}_1 &= -y_1 + 3.4\tanh(y_1) - 1.6\tanh(y_2) + 0.7\tanh(y_3)\\
&\quad + 0.0136\tanh(x_1(t)) - 0.0015\tanh(x_2(t)) + 0.0025\tanh(x_3(t))\\
\dot{y}_2 &= -y_2 + 2.5\tanh(y_1) + 0.95\tanh(y_3)\\
&\quad + 0.0004\tanh(x_1(t)) + 0.0212\tanh(x_2(t)) - 0.0005\tanh(x_3(t))\\
\dot{y}_3 &= -y_3 - 3.5\tanh(y_1) + 0.5\tanh(y_2)\\
&\quad + 0.0012\tanh(x_1(t)) + 0.0023\tanh(x_2(t)) + 0.0145\tanh(x_3(t)).
\end{aligned} \tag{2.6}$$

The network (2.6) is in the form of (2.2), and according to Theorem 2.1, it possesses chaotic motions around the limit cycle of (2.4).

To simulate the results, let us use in HNN (2.6) the chaotic solution $x(t)$ of (2.5) with $x_1(0) = -0.109$, $x_2(0) = -0.832$ and $x_3(0) = 1.721$, and represent the trajectory of (2.6) with $y_1(0) = 0.645$, $y_2(0) = 0.243$ and $y_3(0) = -0.628$ in Fig. 1. The figure supports the result of Theorem 2.1 such that the limit cycle of (2.4) is entrained by the chaos. Moreover, the irregular behavior of the y_3 coordinate over time is illustrated in Fig. 2.

2.1. Sensitivity analysis

The replication of sensitivity in more general coupled systems is rigorously proved in the paper [35]. Here, we will show through simulations the replication of sensitivity by HNNs.

Li et al. [17] theoretically verified the existence of horseshoe chaos in the HNN:

$$\begin{aligned}
\dot{x}_1 &= -x_1 + 2\tanh(x_1) - \tanh(x_2)\\
\dot{x}_2 &= -x_2 + 1.7\tanh(x_1) + 1.71\tanh(x_2) + 1.1\tanh(x_3)\\
\dot{x}_3 &= -2x_3 - 2.5\tanh(x_1) - 2.9\tanh(x_2) + 0.56\tanh(x_3).
\end{aligned} \tag{2.7}$$

Additionally, we consider the HNN

$$\begin{aligned}
\dot{y}_1 &= -y_1 + 3.4\tanh(y_1) - 1.6\tanh(y_2) + 0.7\tanh(y_3)\\
&\quad + 0.02\tanh(x_1(t)) + 0.035\tanh(x_3(t))\\
\dot{y}_2 &= -y_2 + 2.5\tanh(y_1) + 0.95\tanh(y_3) + 0.025\tanh(x_2(t))\\
\dot{y}_3 &= -y_3 - 3.5\tanh(y_1) + 0.5\tanh(y_2) + 0.004\tanh(x_1(t))\\
&\quad - 0.01\tanh(x_2(t)) + 0.05\tanh(x_3(t)),
\end{aligned} \tag{2.8}$$

which is obtained by using of the solutions of (2.7) as external inputs in (2.4).

To demonstrate numerically the replication of sensitivity, we illustrate in Fig. 3 two initially nearby trajectories of the coupled network (2.7)+(2.8), one with the initial data $x_1(0) = 0.236$, $x_2(0) = 0.543$, $x_3(0) = -0.745$, $y_1(0) = -0.751$, $y_2(0) = -0.672$ and $y_3(0) = 1.641$, represented in blue, and the other one with the initial data $x_1(0) = 0.237$, $x_2(0) = 0.541$, $x_3(0) = -0.752$, $y_1(0) = -0.749$, $y_2(0) = -0.674$ and $y_3(0) = 1.643$, pictured in red. Fig. 3(a) and (b) show the projections of these trajectories on the $x_1 - x_2 - x_3$ and $y_1 - y_2 - y_3$ spaces, respectively. It is seen in Fig. 3

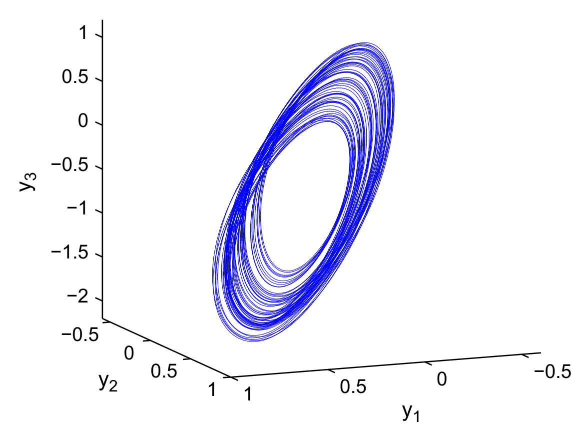

Fig. 1. The chaotic trajectory of HNN (2.6). The figure supports Theorem 2.1 such that the trajectory behaves chaotically around the limit cycle of HNN (2.4).

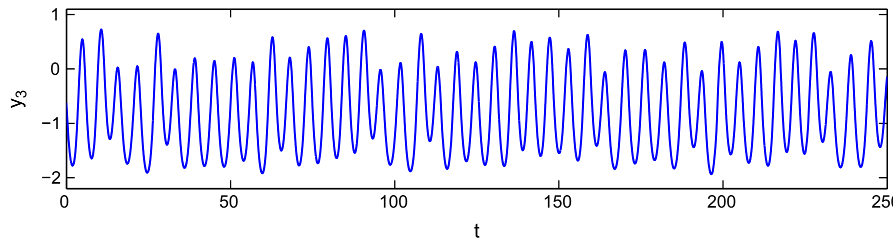

Fig. 2. The chaotic behavior of the y_3 coordinate of HNN (2.6).

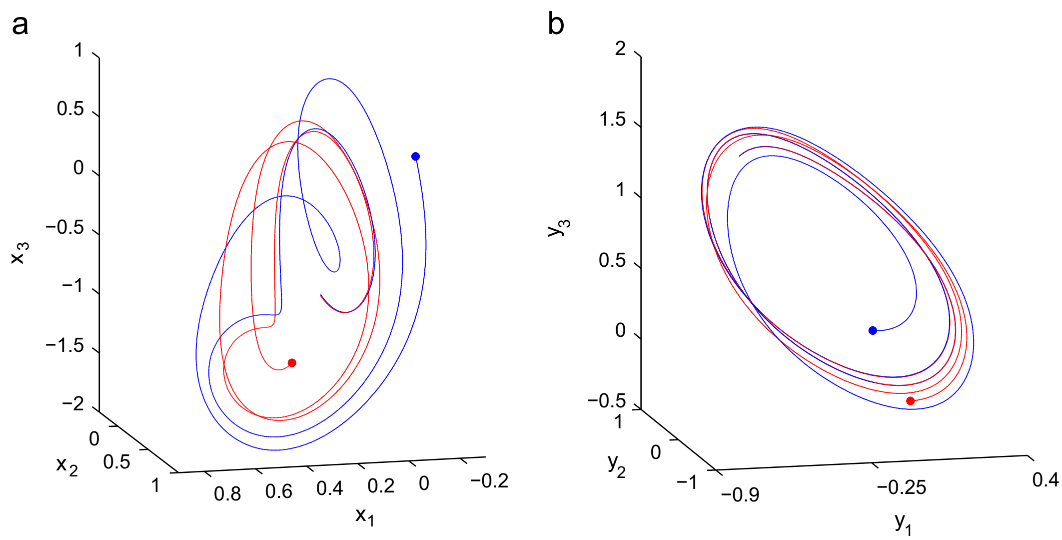

Fig. 3. Extension of sensitivity in the coupled HNNs (2.7) and (2.8). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

(a) that the sensitivity feature is present in the HNN (2.7) such that the initially nearby solutions eventually diverge. On the other hand, it is seen in Fig. 3(b) that the trajectories are initially close to each other and are then separated, that is, the sensitivity is replicated by the network (2.8). The simulations are performed for $t \in [0, 21]$.

2.2. Chaos around tori

Verification of the entrainment of limit tori by chaos is a theoretically difficult task. Nevertheless, in this part of the paper, let us show that near-torus chaos is possible for HNNs. For these needs, similar to the near-limit cycle chaos, we will use the following neural networks.

According to the simulation results of the study [23], the HNN

$$\begin{aligned}
\dot{x}_1 &= -x_1 + \tanh(x_1) + 0.5\tanh(x_2) - 3\tanh(x_3) - \tanh(x_4)\\
\dot{x}_2 &= -x_2 + 2.3\tanh(x_2) + 3\tanh(x_3)\\
\dot{x}_3 &= -x_3 + 3\tanh(x_1) - 3\tanh(x_2) + \tanh(x_3)\\
\dot{x}_4 &= -100x_4 + 100\tanh(x_1) + 170\tanh(x_4)
\end{aligned} \tag{2.9}$$

is hyperchaotic such that it possesses two positive Lyapunov exponents. The chaos will be applied as an input for the following Hopfield neural network:

$$\begin{aligned}
\dot{u}_1 &= -u_1 + \tanh(u_1) + 0.5\tanh(u_2) - 3\tanh(u_3) - \tanh(u_4)\\
\dot{u}_2 &= -u_2 - 0.1\tanh(u_1) + 2\tanh(u_2) + 3\tanh(u_3)\\
\dot{u}_3 &= -u_3 + 3\tanh(u_1) - 3\tanh(u_2) + \tanh(u_3)\\
\dot{u}_4 &= -100u_4 + 100\tanh(u_1) + 170\tanh(u_4).
\end{aligned} \tag{2.10}$$

It is shown in paper [24] that the HNN (2.10) admits the Lyapunov exponents 0, 0, -0.2092 and -46.8691 such that the network possesses a regular torus, which attracts near solutions.

Now, let us perturb the last HNN by solutions of (2.9) as external inputs to obtain

$$\begin{aligned}
\dot{y}_1 &= -y_1 + \tanh(y_1) + 0.5\tanh(y_2) - 3\tanh(y_3)\\
&\quad - \tanh(y_4) + 0.0257\tanh(x_1(t))\\
\dot{y}_2 &= -y_2 - 0.1\tanh(y_1) + 2\tanh(y_2) + 3\tanh(y_3)\\
&\quad + 0.0223\tanh(x_2(t))\\
\dot{y}_3 &= -y_3 + 3\tanh(y_1) - 3\tanh(y_2) + \tanh(y_3) + 0.0159\tanh(x_3(t))\\
\dot{y}_4 &= -100y_4 + 100\tanh(y_1) + 170\tanh(y_4) + 0.0334\tanh(x_4(t)).
\end{aligned} \tag{2.11}$$

Fig. 4 shows the trajectory of (2.11) with $x_1(0) = -0.1321$, $x_2(0) = -0.3589$, $x_3(0) = 0.3914$, $x_4(0) = -1.7219$, $y_1(0) = 0.0259$, $y_2(0) = -0.0096$, $y_3(0) = -0.2383$, and $y_4(0) = -1.5493$. One can see that the motion is chaotic and surrounds the torus. Furthermore, the y_2 coordinate of the solution is represented in Fig. 5. The simulation results reveal that the HNN (2.11) possesses motions that behave chaotically around the torus of (2.10).

2.3. Comparison with synchronization of chaos

The main role of synchronization [73–75] is to predict the properties of the response system or the drive system. Thus, our results may be considered as an indicative of a type of synchronization, if one accepts the following properties to be predicted: the existence of infinitely many unstable periodic solutions with the same periods as those for the drive system, ingredients of chaos, strange attractors, the possibility of controlling chaos, etc.

To analyze our results for generalized synchronization [75–79], we will consider the chaos produced by the networks (2.9) and (2.11). The auxiliary system approach [78,79] as well as the method of conditional Lyapunov exponents [73,76] will be applied to indicate the presence or absence of generalized synchronization in the couple (2.9)+(2.11) considered this time as drive-response systems (as it is accepted in the synchronization theory).

Let us take into account the auxiliary system:

$$\begin{aligned}\dot z_1 &= -z_1+\tanh(z_1)+0.5\tanh(z_2)-3\tanh(z_3)\\ &\quad -\tanh(z_4)+0.0257\tanh(x_1(t))\\ \dot z_2 &= -z_2-0.1\tanh(z_1)+2\tanh(z_2)+3\tanh(z_3)+0.0223\tanh(x_2(t))\\ \dot z_3 &= -z_3+3\tanh(z_1)-3\tanh(z_2)+\tanh(z_3)+0.0159\tanh(x_3(t))\\ \dot z_4 &= -100z_4+100\tanh(z_1)+170\tanh(z_4)+0.0334\tanh(x_4(t)).\end{aligned} \tag{2.12}$$

Making use of the initial data $x_1(0)=-0.1321$, $x_2(0)=-0.3589$, $x_3(0)=0.3914$, $x_4(0)=-1.7219$, $y_1(0)=0.0259$, $y_2(0)=-0.0096$, $y_3(0)=-0.2383$, $y_4(0)=-1.5493$, $z_1(0)=0.1376$, $z_2(0)=-0.0469$, $z_3(0)=0.2524$, and $z_4(0)=1.7589$ and omitting the first 1000 iterations, we obtain the stroboscopic plot of system (2.9)+(2.11)+(2.12) whose projection on the y_2–z_2 plane is shown in Fig. 6. Since the plot is not on the line $z_2=y_2$, we conclude that generalized synchronization does not occur.

Next, to determine the conditional Lyapunov exponents, we consider the following variational equations for the HNN (2.11):

$$\begin{aligned}\dot\eta_1 &= [-1+\mathrm{sech}^2(y_1(t))]\eta_1+0.5\,\mathrm{sech}^2(y_2(t))\eta_2-3\,\mathrm{sech}^2(y_3(t))\eta_3-\mathrm{sech}^2(y_4(t))\eta_4\\ \dot\eta_2 &= -0.1\,\mathrm{sech}^2(y_1(t))\eta_1+[-1+2\,\mathrm{sech}^2(y_2(t))]\eta_2+3\,\mathrm{sech}^2(y_3(t))\eta_3\\ \dot\eta_3 &= 3\,\mathrm{sech}^2(y_1(t))\eta_1-3\,\mathrm{sech}^2(y_2(t))\eta_2+[-1+\mathrm{sech}^2(y_3(t))]\eta_3\\ \dot\eta_4 &= 100\,\mathrm{sech}^2(y_1(t))\eta_1+[-100+170\,\mathrm{sech}^2(y_4(t))]\eta_4.\end{aligned} \tag{2.13}$$

Taking into account the solution $y(t)$ of (2.11) corresponding to the initial data $x_1(0)=-0.1321$, $x_2(0)=-0.3589$, $x_3(0)=0.3914$, $x_4(0)=-1.7219$, $y_1(0)=0.0259$, $y_2(0)=-0.0096$, $y_3(0)=-0.2383$, and $y_4(0)=-1.5493$, we evaluated the largest Lyapunov exponent of system (2.13) as 0.105747. That is, the network (2.11) admits a positive conditional Lyapunov exponent, and this result reveals one more time the absence of generalized synchronization in the coupled HNNs (2.9) and (2.11).

We have shown that the method of extension of chaos by entrainment of tori is not generalized synchronization. This was also affirmed in several other simulations for limit cycles and tori in the paper [35].

3. Extension and control of cyclic/toroidal chaos in neural networks

In the present section, we will apply the instrument of chaos extension to obtain and control chaos in collectives of neural networks. New phenomena of the entrainment of two limit cycles by chaos and attraction of two chaotic cycles by an equilibrium will be demonstrated. Moreover, we will exhibit that the OGY (Ott, Grebogi, Yorke) control method [84] can be applied to stabilize not only periodic motions, but also tori. The control applied to the chaos generating HNNs also affects chaos of the perturbed HNNs. The results of the section may provide new ideas on brain activities, if one takes into account the experimental results in [25,26,33].

3.1. Entrainment of two limit cycles by chaos

We will use one more time the HNN (2.5) as the source of chaotic inputs, but this time the following HNN is to be perturbed by the inputs:

$$\begin{aligned}\dot u_1 &= -u_1+1.5\tanh(u_1)+2.9\tanh(u_2)+0.8\tanh(u_3)\\ \dot u_2 &= -u_2-3.5\tanh(u_1)+1.18\tanh(u_2)\\ \dot u_3 &= -u_3+2.977\tanh(u_1)-22\tanh(u_2)+0.47\tanh(u_3).\end{aligned} \tag{3.1}$$

According to the results of the study [18], the network (3.1) admits two limit cycles with the Lyapunov exponents 0, -0.1792 and

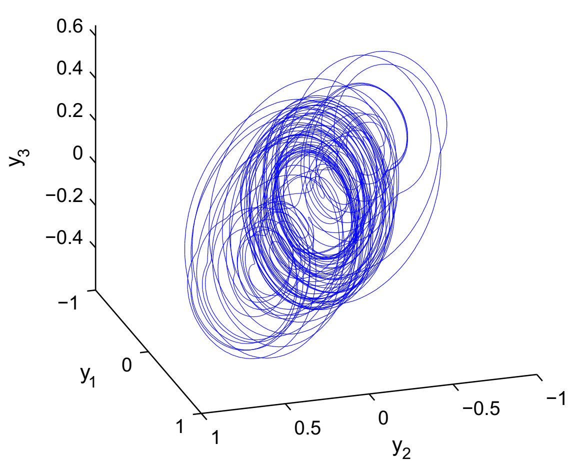

Fig. 4. The chaotic motion around the torus of HNN (2.10).

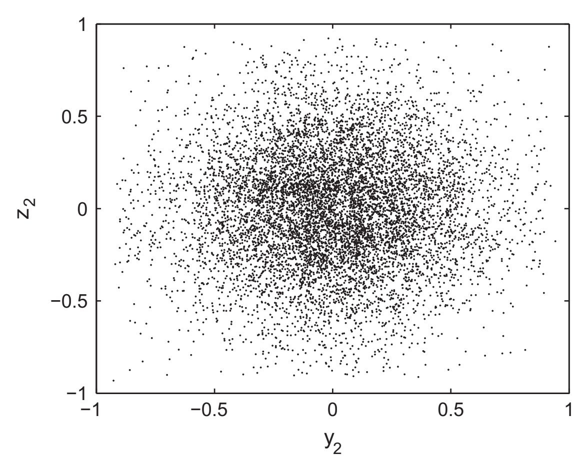

Fig. 6. The auxiliary system approach applied to the coupled HNNs (2.9) and (2.11).

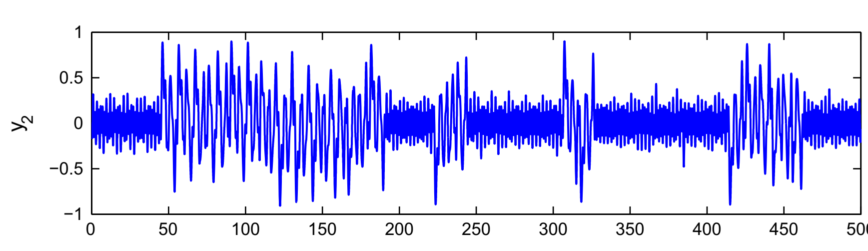

Fig. 5. The graph of the y_2 coordinate of HNN (2.11).

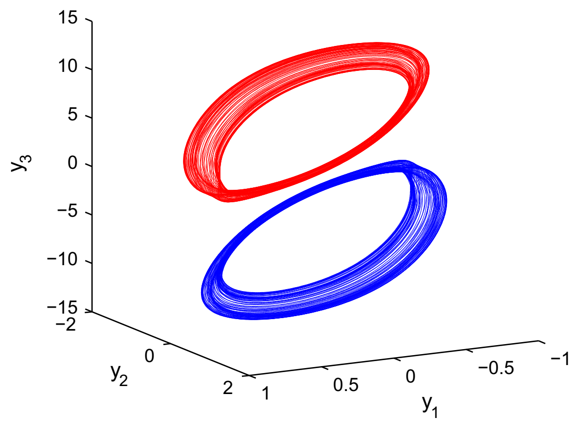

Fig. 7. Entrainment of two limit cycles by chaos. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

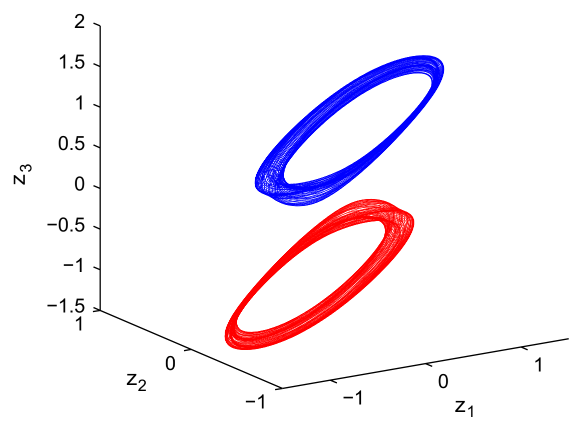

Fig. 8. Attraction of two chaotic cycles by an equilibrium. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

-0.7083 such that the cycles are orbitally stable by the Andronov–Witt Theorem [82].

Beside the last equations, consider the following HNN:

$$\begin{aligned}
\dot{y}_1 &= -y_1+1.5\tanh(y_1)+2.9\tanh(y_2)+0.8\tanh(y_3)\\
&\quad +0.04\tanh(x_1(t))+0.03\tanh(x_2(t)) \quad +0.006\tanh(x_3(t))\\
\dot{y}_2 &= -y_2-3.5\tanh(y_1)+1.18\tanh(y_2)-0.002\tanh(x_1(t))\\
&\quad +0.06\tanh(x_2(t))\\
\dot{y}_3 &= -y_3+2.977\tanh(y_1)-22\tanh(y_2)+0.47\tanh(y_3)\\
&\quad -0.001\tanh(x_2(t))+0.04\tanh(x_3(t)). \qquad (3.2)
\end{aligned}$$

By localizing the result of Theorem 2.1 near the two limit cycles, one can conclude that (3.2) admits two chaotic cycles. Fig. 7 represents the trajectories of (3.2) corresponding to the initial data $x_1(0)=1.903$, $x_2(0)=0.221$, $x_3(0)=-4.011$, $y_1(0)=0.713$, $y_2(0)=0.273$, and $y_3(0)=-10.001$ and $x_1(0)=-0.532$, $x_2(0)=-1.647$, $x_3(0)=2.607$, $y_1(0)=0.571$, $y_2(0)=0.117$, and $y_3(0)=-0.079$ shown in blue and red colors, respectively. One can see in Fig. 7 that two chaotic cycles appear in the dynamics of the network (3.2). We call this phenomenon as the entrainment of two limit cycles by chaos.

One can predict that the appearance of cyclic chaos can be implemented for HNNs with not only two cycles, but also several ones. Moreover, the chaos extension by the entrainment procedure can be realized for different types of neural networks.

3.2. Attraction of two chaotic cycles by an equilibrium

In our paper [36], we considered extension of chaos in neighborhoods of attracting equilibria. From the simple observation for a dynamical system that a periodic solution used as a perturbation may cause a new cycle under certain conditions, one can conclude that near-limit cycle chaotic inputs can lead to similar outputs for systems with stable equilibria. To verify this hypothesis numerically, let us apply the double cyclic chaos obtained for system (3.2) as an input to the HNN

$$\begin{aligned}
\dot{u}_1 &= -u_1+0.005\tanh(u_1))+0.009\tanh(u_2)-0.008\tanh(u_3)\\
\dot{u}_2 &= -u_2-0.001\tanh(u_1)+0.007\tanh(u_2)-0.003\tanh(u_3)\\
\dot{u}_3 &= -u_3+0.009\tanh(u_1)-0.002\tanh(u_2)+0.004\tanh(u_3), \qquad (3.3)
\end{aligned}$$

which admits the primitive asymptotically stable solution, to set up the HNN

$$\begin{aligned}
\dot{z}_1 &= -z_1+0.005\tanh(z_1))+0.009\tanh(z_2)-0.008\tanh(z_3)\\
&\quad +4\tanh(y_1(t))+\tanh(y_2(t))\\
\dot{z}_2 &= -z_2-0.001\tanh(z_1)+0.007\tanh(z_2)-0.003\tanh(z_3)\\
&\quad +0.5\tanh(y_1(t))+2\tanh(y_2(t)) \quad +0.5\tanh(y_3(t))\\
\dot{z}_3 &= -z_3+0.009\tanh(z_1)-0.002\tanh(z_2)+0.004\tanh(z_3)\\
&\quad +2\tanh(y_1(t))-\tanh(y_3(t)). \qquad (3.4)
\end{aligned}$$

Fig. 8 shows the simulation results such that two chaotic cycles appear in the dynamics of the network (3.4). In the simulation, we used the solutions of (3.2) represented in Fig. 7, and depicted with the corresponding same colors in Fig. 8 the trajectories of (3.4) with the initial data $z_1(0)=0.705$, $z_2(0)=0.487$, $z_3(0)=0.997$ and $z_1(0)=-0.142$, $z_2(0)=0.408$, $z_3(0)=-0.873$ in blue and red, respectively.

3.3. OGY control of a torus

The control of chaos in neural networks is supposed to be the reason for the appearance of limit cycles in the experiments of neurobiologists [25,26]. We showed in [35] how the Pyragas control method [47] stabilizes entrained limit cycles. It is easy to see that the simulations can be adapted for neural networks in the form of (2.5)+(2.6). In this subsection, we will demonstrate a novel application of the OGY control [83,84] to stabilize tori. Since the OGY control is for discrete equations, we will start with the description of piecewise constant perturbations, which will be controlled by the method.

Consider the function

$$P(t,\theta)=\begin{cases}1.6 & \text{if } \theta_{2i}<t\le\theta_{2i+1},\\ 0.2 & \text{if } \theta_{2i+1}<t\le\theta_{2i+2},\end{cases} \qquad (3.5)$$

where i is a nonnegative integer, the sequence $\theta=\{\theta_i\}$ is defined through the equation $\theta_i=i+\zeta_i$ with $\zeta_{i+1}=F_\lambda(\zeta_i)$, $\zeta_0\in[0,1]$, and $F_\lambda(u)=\lambda u(1-u)$ is the logistic map. The map $F_\lambda(u)$ is chaotic through period-doubling cascade for $\lambda=3.8$, and the interval $[0,1]$ is invariant under its iterations [83].

Let us describe the OGY control method for the logistic map [83]. Suppose that the parameter λ in the map $F_\lambda(u)$ is allowed to vary in the range $[3.8-\varepsilon, 3.8+\varepsilon]$, where ε is a given small positive number. Consider an arbitrary solution $\{\zeta_i\}$, $\zeta_0\in[0,1]$, of the map and denote by $\zeta^{(j)}$, $j=1,2,\dots,p$, the target $p-$ periodic orbit to be stabilized. In the control procedure [83,84], at each iteration step i after the control mechanism is switched on, we consider the logistic map with the parameter value $\lambda=\overline{\lambda}_i$, where

$$\overline{\lambda}_i = 3.8\left(1+\frac{(2\zeta^{(j)}-1)(\zeta_i-\zeta^{(j)})}{\zeta^{(j)}(1-\zeta^{(j)})}\right), \tag{3.6}$$

provided that the number on the right-hand side of the formula (3.6) belongs to the interval $[3.8-\varepsilon, 3.8+\varepsilon]$. In other words, formula (3.6) is valid if the trajectory $\{\zeta_i\}$ is sufficiently close to the target periodic orbit. Otherwise, we take $\overline{\lambda}_i = 3.8$ so that the system evolves at its original parameter value, and wait until the trajectory $\{\zeta_i\}$ enters in a sufficiently small neighborhood of the periodic orbit $\zeta^{(j)}$, $j=1,2,\ldots,p$, such that the inequality $-\varepsilon \le 3.8(2\zeta^{(j)}-1)(\zeta_i-\zeta^{(j)})/[\zeta^{(j)}(1-\zeta^{(j)})] \le \varepsilon$ holds. If this is the case, the chaos control is not achieved immediately after switching on the control mechanism. Instead, there is a transition time before the desired periodic orbit is stabilized. The transition time increases if the number ε decreases [79].

Let us introduce the Hopfield neural network

$$\begin{aligned}
\dot{x}_1 &= -7x_1 + 0.012\tanh(x_1) - 0.016\tanh(x_2) + 0.003\tanh(x_3)\\
\dot{x}_2 &= -4x_2 - 0.004\tanh(x_1) + 0.013\tanh(x_2)\\
&\quad + 0.005\tanh(x_3) + P(t,\theta)\\
\dot{x}_3 &= -6x_3 + 0.008\tanh(x_1) + 0.005\tanh(x_2)\\
&\quad + 0.009\tanh(x_3) + \sin(4t) + P(t,\theta),
\end{aligned} \tag{3.7}$$

where the piecewise constant function $P(t,\theta)$ described by (3.5) is used as a chaotic input. Since the functions $P(t,\theta)$ and $\sin(4t)$ lead to the presence of infinitely many quasi-periodic inputs with incommensurate periods, multiples of 2 and $\pi/2$, respectively, one can use the results of [85–88] to conclude that the HNN (3.7) with $\lambda=3.8$ possesses a chaotic attractor with infinitely many unstable quasi-periodic solutions.

Using the solutions of (3.7) as inputs for the HNN

$$\begin{aligned}
\dot{u}_1 &= -3u_1 + 0.003\tanh(u_1) - 0.005\tanh(u_2) - 0.013\tanh(u_3)\\
\dot{u}_2 &= -8u_2 + 0.007\tanh(u_1) + 0.008\tanh(u_2) + 0.007\tanh(u_3)\\
\dot{u}_3 &= -6u_3 - 0.004\tanh(u_1) - 0.006\tanh(u_2) + 0.002\tanh(u_3),
\end{aligned} \tag{3.8}$$

we set up the network

$$\begin{aligned}
\dot{y}_1 &= -3y_1 + 0.003\tanh(y_1) - 0.005\tanh(y_2) - 0.013\tanh(y_3) + 1.5x_1(t)\\
\dot{y}_2 &= -8y_2 + 0.007\tanh(y_1) + 0.008\tanh(y_2) + 0.007\tanh(y_3) + 1.8x_2(t)\\
\dot{y}_3 &= -6y_3 - 0.004\tanh(y_1) - 0.006\tanh(y_2) + 0.002\tanh(y_3) + 1.2x_3(t).
\end{aligned} \tag{3.9}$$

It is worth noting that the origin is the asymptotically stable equilibrium point of (3.8). According to the results of the study [36], the network (3.9) possesses a chaotic attractor with infinitely many unstable quasi-periodic solutions, provided that the value $\lambda=3.8$ is used in (3.7).

The trajectories of (3.7) and (3.9) with $\lambda=3.8$ corresponding to the initial data $x_1(t_0)=-0.0007$, $x_2(t_0)=0.3983$, $x_3(t_0)=0.2061$, $y_1(t_0)=-0.0004$, $y_2(t_0)=0.0801$, and $y_3(t_0)=0.0487$, where $t_0=0.281$, are represented in Fig. 9(a) and (b), respectively. Moreover, the graphs of the x_3 and y_3 coordinates of the same trajectories are depicted in Fig. 10. The simulations reveal that both of the HNNs (3.7) and (3.9) exhibit chaotic motions.

Next, we consider the solution of the coupled network (3.7)+(3.9) with the same initial data as considered in Figs. 9 and 10, and apply the OGY control method around the fixed point 2.8/3.8 of the logistic map $F_{3.8}(u)$. Fig. 11 shows the simulation results for the x_3 and y_3 coordinates. The value $\varepsilon=0.06$ is used in the simulation. The control mechanism is switched on at $t=\theta_{30}$ and switched off at $t=\theta_{60}$. The control becomes dominant approximately at $t=45$ and its effect lasts approximately until $t=115$, after which the instability becomes dominant and irregular behavior develops again. It is seen that a quasi-periodic solution of the HNN (3.7) is stabilized, and accordingly, the chaos of the HNN (3.9) is controlled by the stabilization of the corresponding quasi-periodic solution. On the other hand, Fig. 12(a) and (b) represent the stabilized tori of the networks (3.7) and (3.9), respectively.

4. Conclusions

We have provided theoretical arguments for the entrainment of limit cycles and tori by chaos in neural networks by applying basic Hopfield neural networks. In Section 3, several opportunities of the chaos extension are considered when the number of limit cycles varies, and we performed the attraction of chaotic cycles by equilibria. All these demonstrate the potentials of our approach, which can be realized in the theory of neural networks.

It is natural to suppose that instead of a unique limit cycle or near-limit cycle chaos as it was the case in the experiments [33,34,38–40], one and the same stimulus may cause the presence of several such behaviors if the experiments are performed intentionally. Our dynamical results support this idea, and they can be developed easily in the mathematical sense (and hopefully in brain behavior researches) for various numbers and types of stimuli as well as chaotic and regular outputs. We suggest that not only limit cycles and near-limit cycle chaos but also limit tori and near-limit tori chaos can be investigated in experiments. Another possible experimental program concerning our results is to follow the papers [2,3,25,26], where it was claimed that the memory capacity depends strongly on chaos. Loosely speaking, complexity

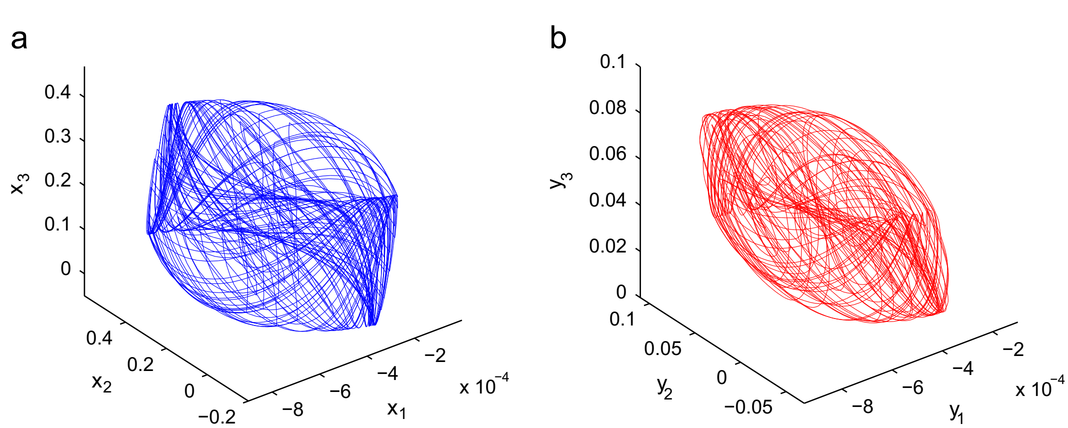

Fig. 9. The chaotic trajectories of (3.7) and (3.9) are represented in (a) and (b), respectively.

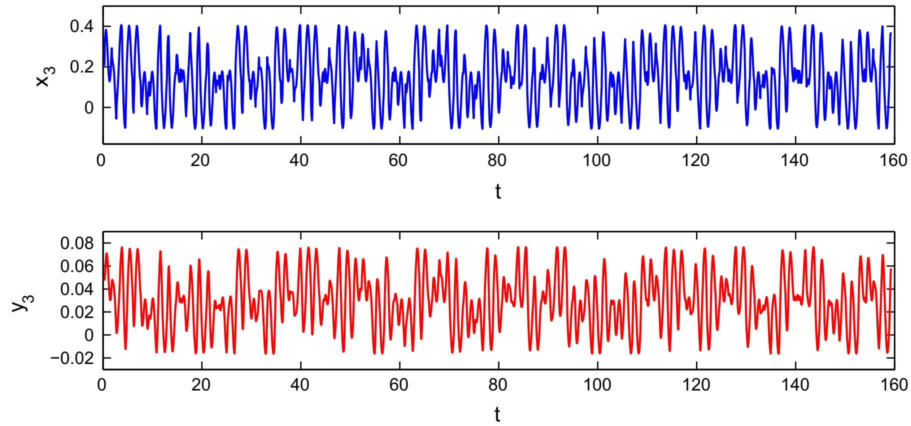

Fig. 10. The chaotic behavior in the networks (3.7) and (3.9).

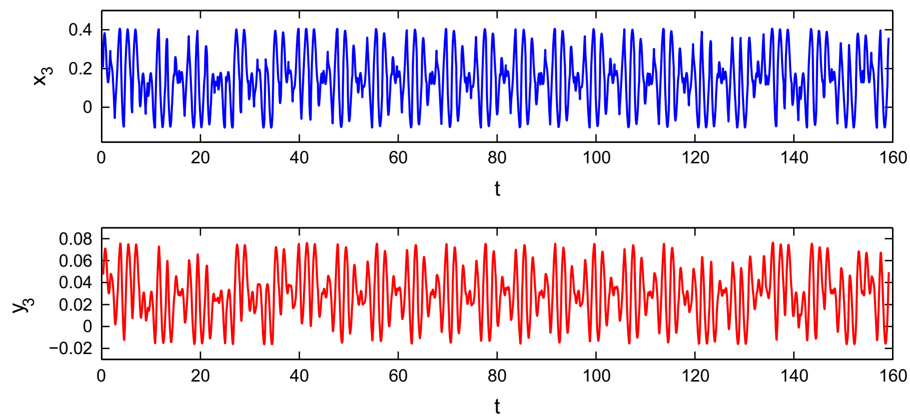

Fig. 11. The application of the OGY control method to stabilize the quasi-periodic solutions of (3.7) and (3.9). The control is switched on at $t=\theta_{30}$ and switched off at $t=\theta_{60}$. The value $\varepsilon=0.06$ is used in the control procedure.

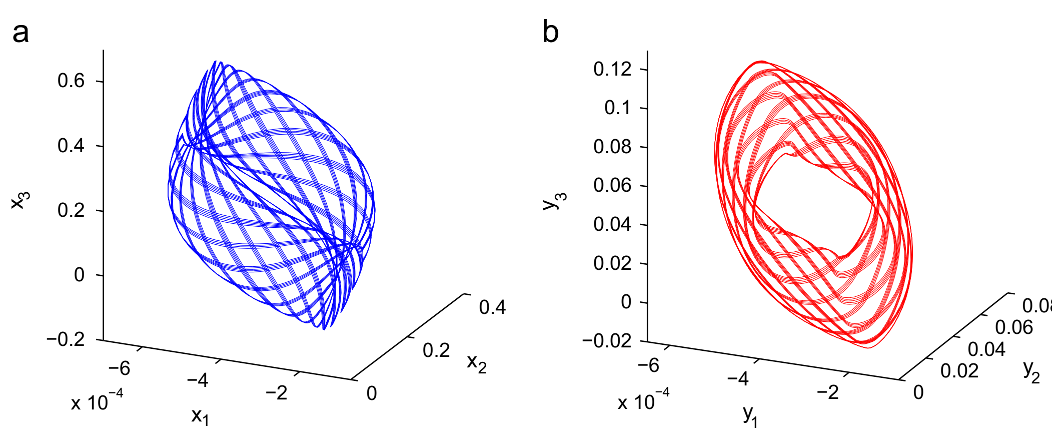

Fig. 12. The stabilized tori of HNNs (3.7) and (3.9) are shown in (a) and (b), respectively.

of behavior, its degree of irregularity, is proportional to the memory capacity. It is obvious that to have a larger memory, we have to make chaos more "complex". For example, it is known that periodic solutions (unstable) are in the basis of Li-Yorke and Devaney chaos [41,42]. By replacing the periodic motions with quasi-periodic or even almost periodic ones [44], we have more complex chaos.

In papers [25,26,34], the limit cycle appearance in the chaotic set of motions was mentioned without an explicit indication of the reason for the phenomenon. One can suspect that this is because of the chaos control [47,84], that is, the stabilization of periodic solutions. However, the control procedure uses a special mechanism in which the solutions are involved [47,84]. One can suggest that external stimuli are not controlling the cycle, but just trigger the control mechanism in neural networks. From this point of view, we have to say that our results extend the comprehension of the mechanism from limit cycles to tori. Moreover, we develop the idea that the control of chaos applied to a certain neural network can be extended to those which are adjoint with the controlled one. The extension of chaos control may give some positive information for the synchronized behavior in large society of neural networks to govern a motion of human body.

Another process in brain behavior that our results concern with is the synchronization of neural networks. Since chaos is an attribute of neural networks and synchronization is necessary for the effective brain work, one should say about synchronization of chaos in neural networks. For the moment, the most developed one is the generalized synchronization [75–79], which requests asymptotic closeness of drive and response systems. In Section 2.3, it is proved by the method of auxiliary system approach and conditional Lyapunov exponents applied to coupled (2.9)+(2.11) that the presented method is not generalized synchronization. In other words, there are not necessarily asymptotic relations between the two networks. Moreover, our method reveals that the systems participating in the extension of chaos are synchronized in the sense that chaos may admit similar properties such as the presence of motions with the same periods, similarity of chaotic attractors and bifurcation diagrams, property to be controlled simultaneously, Shilnikov orbits, and intermittency [35–37,85–88]. Thus, the present results may be useful for neurobiologists to give more directions as well as mathematical apparatus for the future joint investigations.

Chaotic itinerancy [89] is a universal dynamics in high-dimensional systems, showing itinerant motion among varieties of low-dimensional ordered states through high-dimensional chaos. This phenomenon occurs in non-equilibrium neural networks [10] and analysis of brain activities [33]. In its degenerated form, chaotic itinerancy is related to intermittency [90] since both of them represent dynamical interchange of irregularity and regularity. Likewise the itinerant chaos observed in brain activities, low-dimensional chaos occurs in our results, and high-dimensional chaos takes place when all subsystems are considered as a whole. The main difference between our technique and chaotic itinerancy is in the elapsed time for the occurrence of the processes. No itinerant motion is observable in our discussions and all resultant chaotic subsystems process simultaneously, whereas the low-dimensional chaotic motions take place as time elapses in the case of chaotic itinerancy. The knowledge of the chaos type is another difference between chaotic itinerancy and our approach [36,37,85–87].

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