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# HOPF BIFURCATION FOR A 3D FILIPPOV SYSTEM

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**Abstract.** We study the behaviour of solutions for a 3-dimensional system of differential equations with discontinuous right hand side in the neighbourhood of the origin. Using B-equivalence of that system to an impulsive differential equation [3, 4], existence of a center manifold is proved, and then a Hopf bifurcation theorem is provided for such equations in the critical case. The results are apparently obtained for the systems with dimensions greater than two for the first time. Finally, an appropriate example is given to illustrate our results.
**Keywords.** Filippov systems, Impulsive differential equations, Center manifold, Hopf bifurcation.
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## 1 Introduction

When we consider bifurcations of a given type in a neighborhood of the origin, the center manifold theory appears as one of the most effective tools in the investigation. The study of center manifolds can be traced back to the works of Pliss [19] and Kelley [11]. When such manifolds exist, the investigation of local behaviours can be reduced to the study of the systems on the center manifolds. Any bifurcations which occur in the neighborhood of the origin on the center manifold are guaranteed to occur in the full nonlinear system as well. In particular, if a limit cycle exists on the center manifold, then it will also appear in the full system.

Physical phenomena are often modeled by discontinuous dynamical systems which switch between different vector fields in different modes. Filippov systems form a subclass of discontinuous systems described by differential equations with a discontinuous right-hand side [9]. Bifurcations in smooth systems are well understood, but little is known in discontinuous dynamical systems. In the last several decades, existence of non-smooth dynamics in the real world has stimulated the study of bifurcation of periodic solutions in discontinuous systems [8, 10], [12] – [18]. Furthermore, Bautin and Leontovich

[7] and Küpper et al. [14, 15] have considered Hopf bifurcation for planar Filippov systems with discontinuities on a single straight line. However, to the best of our knowledge, there have been no results considering bifurcation in three and more dimensions for equations with discontinuous vector fields.

In [2], Hopf bifurcation has been investigated for planar discontinuous dynamical systems. Based on the method of $B-$equivalence [1]−[5] to impulsive differential equations and by using the projection on the center manifold, we extend the results in [2] to obtain qualitative properties for our three dimensional system with discontinuous right-hand side. The present paper deals with discontinuities on arbitrarily finite nonlinear surfaces. In fact, it is the advantage of the $B-$equivalence method that we can consider a system with nonlinear discontinuity sets.

The structure of the paper is as follows. Section 2 describes the nonperturbed system and studies its qualitative properties. Section 3 is dedicated to the perturbed system and the notion of $B-$equivalent impulsive systems. The center manifold theory is given in Section 4. Our main results concerning the bifurcation of periodic solutions are formulated in Section 5. In the last section, we present an appropriate example to illustrate our findings.

## 2 The nonperturbed system

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively. Let $\mathbb{R}^n$, $n \in \mathbb{N}$, be the n-dimensional real space and $\langle x, y\rangle$ denote the scalar product for all vectors $x, y \in \mathbb{R}^n$. The norm of a vector $x \in \mathbb{R}^n$ is given by $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$.

Also for the sake of brevity in the sequel, every angle for a point is considered with respect to the positive half-line of the first coordinate axis in $x_1x_2-$plane. Moreover, it is important to note that we shall consider angle values only in the interval $[0, 2\pi]$ because of the periodicity.

Before introducing the nonperturbed system, we give the following assumptions and notations which will be needed throughout the paper:

(A1) Let $\{\mathcal{P}_i\}_{i=1}^p$, $p \geq 2$, $p \in \mathbb{N}$, be a set of half-planes starting at the $z-$axis, i.e., $\mathcal{P}_i = l_i \times \mathbb{R}$, where $l_i$ are half-lines which start at the origin and are given by $\varphi_i(x) = 0$, $\varphi_i(x) = \langle a^i, x\rangle$, $x \in \mathbb{R}^2$ and $a^i = ({a_1}^i, {a_2}^i) \in \mathbb{R}^2$ are constant vectors (see Fig. 1). Let $\gamma_i$ denote the angle of the line $l_i$ for each $i = \overline{1,p}$ such that

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_p < 2\pi\,;$$

(A2) There exist constant, real-valued $2 \times 2$ matrices $A_i$ defined by $A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}$ where $\beta_i > 0$ and constants $b_i \in \mathbb{R}$, $i = \overline{1,p}$.

(N1) $\theta_1 = (2\pi + \gamma_1) - \gamma_p$ and $\theta_i = \gamma_i - \gamma_{i-1}$, $i = \overline{2,p}\,$;

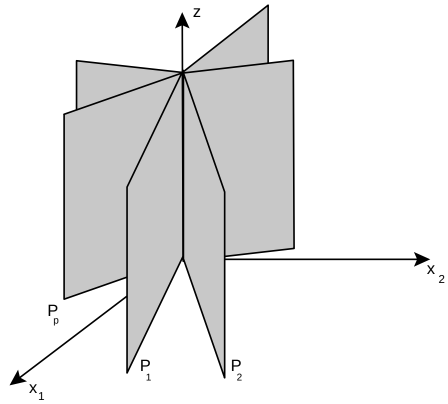


Figure 1: Half-planes $\mathcal{P}_i$, $i = \overline{1,p}$, of discontinuities for the nonperturbed system (1).

(N2) Let $D_i$ denote the region situated between the planes $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$ and defined in cylindrical coordinates $(r, \phi, z)$, where $x_1 = r\cos\phi$, $x_2 = r\sin\phi$ and $z = z$, by

$$D_1 = \{(r, \phi, z) \mid r \geq 0,\ \gamma_p < \phi \leq \gamma_1 + 2\pi,\ z \in \mathbb{R}\},$$

$$D_i = \{(r, \phi, z) \mid r \geq 0,\ \gamma_{i-1} < \phi \leq \gamma_i,\ z \in \mathbb{R}\},\ i = \overline{2,p}.$$

Under the assumptions made above, we study in $\mathbb{R}^3$ the following non-perturbed system,

$$\begin{aligned} \frac{dx}{dt} &= F(x), \\ \frac{dz}{dt} &= f(z), \end{aligned} \tag{1}$$

where $F(x) = A_i x$ and $f(z) = b_i z$ for $(x, z) \in D_i$, $i = \overline{1,p}$.

We note that the functions $F$ and $f$ in system (1) are discontinuous on the planes $\mathcal{P}_i$, $i = \overline{1,p}$.

*Remark 2.1 It follows from the assumptions* (A1) *and* (A2) *that*

$$\langle \frac{\partial \varphi_i(x)}{\partial x}, F(x) \rangle \neq 0 \quad \text{for } x \in l_i,\ i = \overline{1,p}.$$

*That is, the vector field is transversal at every point on $\mathcal{P}_i$ for each $i$.*

Since the results can be most conveniently stated in terms of cylindrical coordinates, we use the transformation $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $z = z$ so that system (1) reduces to

$$\begin{aligned} \frac{dr}{d\phi} &= G(r), \\ \frac{dz}{d\phi} &= g(z), \end{aligned} \tag{2}$$

where $G(r) = \lambda_i r$ and $g(z) = k_i z$ if $(r, \phi, z) \in D_i$, with $\lambda_i = \dfrac{\alpha_i}{\beta_i}$ and $k_i = \dfrac{b_i}{\beta_i}$, $i = \overline{1,p}$. We see that the functions $G$ and $g$ given in (2) have discontinuities when $\phi = \gamma_i$, $i = \overline{1,p}$.

The solution $(r(\phi, r_0), z(\phi, z_0))$ of (2) starting at the point $(0, r_0, z_0)$ is given by

$$r(\phi, r_0) = \begin{cases} \exp(\lambda_1 \phi)\, r_0, & \text{if} \quad 0 \le \phi \le \gamma_1, \\ \exp\{\lambda_1\gamma_1 + \lambda_2\theta_2 + \cdots + \lambda_i(\phi - \gamma_{i-1})\} r_0, & \text{if} \quad \gamma_{i-1} < \phi \le \gamma_i, \\ \exp\{\lambda_1[\phi - (\gamma_p - \gamma_1)] + \sum_{i=2}^{p} \lambda_i\theta_i\} r_0, & \text{if} \quad \gamma_p < \phi \le 2\pi, \end{cases}$$

$$z(\phi, z_0) = \begin{cases} \exp(k_1 \phi) z_0, & \text{if} \quad 0 \le \phi \le \gamma_1, \\ \exp\{k_1\gamma_1 + k_2\theta_2 + \cdots + k_i(\phi - \gamma_{i-1})\} z_0, & \text{if} \quad \gamma_{i-1} < \phi \le \gamma_i, \\ \exp\{k_1[\phi - (\gamma_p - \gamma_1)] + \sum_{i=2}^{p} k_i\theta_i\} z_0, & \text{if} \quad \gamma_p < \phi \le 2\pi, \end{cases}$$

for $i = 2, 3, \ldots, p$.

Now, we define a section $\mathsf{P} = \{(x_1, x_2, z) \mid x_2 = 0, x_1 > 0, z \in \mathbb{R}\}$. Constructing the Poincaré return map on $\mathsf{P}$, we find that

$$(r(2\pi, r_0), z(2\pi, z_0)) = (\exp(\sum_{i=1}^{p} \lambda_i\theta_i) r_0, \exp(\sum_{i=1}^{p} k_i\theta_i) z_0).$$

Let us denote

$$q_1 = \exp(\sum_{i=1}^{p} \lambda_i\theta_i), \tag{3}$$

$$q_2 = \exp(\sum_{i=1}^{p} k_i\theta_i). \tag{4}$$

Since $r(2\pi, r_0) = q_1 r_0$, $z(2\pi, z_0) = q_2 z_0$, we can establish the following assertions.

Lemma 2.1 *Assume that* $q_1 = 1$. *If*

*(i)* $q_2 = 1$, *then all solutions are periodic with period* $T = \sum_{i=1}^{p} \frac{\theta_i}{\beta_i}$, *i.e.,* $\mathbb{R}^3$ *is a center manifold;*

*(ii)* $q_2 < 1$, *then a solution that starts to its motion on* $x_1x_2-$*plane is* $T-$*periodic and all other solutions lie on the surface of a cylinder and they move toward the* $x_1x_2-$*plane, i.e.,* $x_1x_2-$*plane is a center manifold and* $z-$*axis is a stable manifold;*

*(iii)* $q_2 > 1$, *then a solution that starts to its motion on* $x_1x_2-$*plane is* $T-$*periodic and all other solutions lie on the surface of a cylinder and they move away from the origin, i.e.,* $x_1x_2-$*plane is a center manifold and* $z-$*axis is an unstable manifold.*

LEMMA 2.2 *Assume that* $q_1 < 1$. *If*

*(i)* $q_2 = 1$, *then a solution that starts to its motion on* $z-$*axis is* $T-$*periodic and all other solutions will approach to* $z-$*axis, i.e.,* $x_1x_2-$*plane is a stable manifold and* $z-$*axis is a center manifold;*

*(ii)* $q_2 < 1$, *all solutions will spiral toward the origin, i.e., the origin is asymptotically stable;*

*(iii)* $q_2 > 1$, *a solution that starts to its motion on* $x_1x_2-$*plane spirals toward the origin and a solution initiating on* $z-$*axis will move away from the origin, i.e.,* $x_1x_2-$*plane is a stable manifold and* $z-$*axis is a center manifold.*

LEMMA 2.3 *Assume that* $q_1 > 1$. *If*

*(i)* $q_2 = 1$, *then a solution that starts to its motion on* $z-$*axis is* $T-$*periodic and all other solutions move away from the* $z-$*axis, i.e.,* $x_1x_2-$*plane is an unstable manifold and* $z-$*axis is a center manifold;*

*(ii)* $q_2 < 1$, *a solution that starts to its motion on* $x_1x_2-$*plane moves away from the origin and a solution initiating on* $z-$*axis spirals toward the origin, i.e.,* $x_1x_2-$*plane is an unstable manifold and* $z-$*axis is a stable manifold;*

*(iii)* $q_2 > 1$, *all solutions move away from the origin, i.e., the origin is unstable.*

*Remark 2.2 From now on, we assume that* $q_1 = 1$ *and* $q_2 < 1$. *In other words,* $x_1x_2-$*plane is a center manifold and* $z-$*axis is a stable manifold.*

# 3 The perturbed system

Let $\Omega \subset \mathbb{R}^3$ be a domain in the neighborhood of the origin. The following conditions are assumed to hold throughout the paper.

(P1) Let $\{\mathcal{S}_i\}_{i=1}^p$, $p \geq 2$, be a set of cylindrical surfaces which start at the $z-$axis, i.e., $\mathcal{S}_i = c_i \times \mathbb{R}$, where $c_i$ are curves starting at the origin and determined by the equations $\tilde{\varphi}_i(x) = 0$, $\tilde{\varphi} = \langle a^i, x \rangle + \tau_i(x)$, $x \in \mathbb{R}^2$, $\tau_i(x) = o(\|x\|)$ and the constant vectors $a^i$ are the same as described in $(A1)$.

Without loss of generality, we may assume that $\gamma_i \neq \frac{\pi}{2} j$, $j = 1, 3$. Using the transformation $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, equation of the curve $c_i$ can be written, for sufficiently small $r$, as follows [2]

$$c_i : \phi = \gamma_i + \psi_i(r, \phi), \; i = \overline{1, p}, \tag{5}$$

where $\psi_i$ is a $2\pi-$periodic function in $\phi$, continuously differentiable and $\psi_i = O(r)$. Then, we can define the region situated between the surfaces $\mathcal{S}_{i-1}$ and $\mathcal{S}_i$ as follows:

$$\tilde{D}_1 = \{(r, \phi, z) \mid r \geq 0, \; \gamma_p + \psi_p(r, \phi) < \phi \leq \gamma_1 + 2\pi + \psi_1(r, \phi), \; z \in \mathbb{R}\},$$

$\tilde{D}_i = \{(r, \phi, z) \mid r \geq 0, \; \gamma_{i-1} + \psi_{i-1}(r, \phi) < \phi \leq \gamma_i + \psi_i(r, \phi), \; z \in \mathbb{R}\}$, where $i = \overline{2, p}$.

Let $\varepsilon$ be a positive number and $N_\varepsilon(\tilde{D}_i)$ denote the $\varepsilon-$neighborhoods of the regions $\tilde{D}_i$, $i = \overline{1, p}$. In addition to $(P1)$, we assume the following list of conditions.

(P2) Let $f_i$, $h_i$, $i = \overline{1, p}$, be functions defined on $N_\varepsilon(\tilde{D}_i)$ and satisfy $f_i$, $h_i \in C^{(2)}(N_\varepsilon(\tilde{D}_i))$;

(P3) $\tau_i \in C^{(2)}(N_\varepsilon(\tilde{D}_i))$, $i = \overline{1, p}$;

(P4) $f_i(x, z) = o(\|x, z\|)$, $h_i(x, z) = o(\|x, z\|)$, and $f_i(0, z) = 0$, $h_i(0, z) = 0$ for all $z \in \mathbb{R}$, $i = \overline{1, p}$.

We define for $(x, z) \in \tilde{D}_i$, two functions by $\tilde{F}(x, z) = A_i x + f_i(x, z)$ and $\tilde{f}(x, z) = b_i z + h_i(x, z)$, where the matrix $A_i$ and the constant $b_i$ are as defined in $(A2)$ above. In the neighborhood $\Omega$, we consider the following system

$$\begin{aligned} \frac{dx}{dt} &= \tilde{F}(x, z), \\ \frac{dz}{dt} &= \tilde{f}(x, z). \end{aligned} \tag{6}$$

Here, it can be easily seen that the functions $\tilde{F}(x, z)$ and $\tilde{f}(x, z)$ have discontinuities on the surfaces $\mathcal{S}_i$, $i = \overline{1, p}$.

For sufficiently small neighborhood $\Omega$, it follows from the conditions $(A1)$ and $(P1)$ that the surfaces $\mathcal{S}_i$ intersect each other only at $z-$axis, none of them can intersect itself and $\langle \frac{\partial \tilde{\varphi}_i(x)}{\partial x}, \tilde{F}(x, 0) \rangle \neq 0$ for $x \in c_i$, $i = \overline{1, p}$.

If a solution of system (6) starts at a point, which is sufficiently close to the origin and on the surface $\mathcal{S}_i$ with fixed $i$, then this solution can be continued either to the surface $\mathcal{S}_{i+1}$ or $\mathcal{S}_{i-1}$ depending on the direction of the time.

We make use of cylindrical coordinates and rewrite the system (6) in the following equivalent form

$$\begin{aligned} \frac{dr}{d\phi} &= \tilde{G}(r, \phi, z), \\ \frac{dz}{d\phi} &= \tilde{g}(r, \phi, z), \end{aligned} \tag{7}$$

where $\tilde{G}(r, \phi, z) = \lambda_i r + P_i(r, \phi, z)$ and $\tilde{g}(r, \phi, z) = k_i z + Q_i(r, \phi, z)$ whenever $(r, \phi, z) \in \tilde{D}_i$. The functions $P_i$ and $Q_i$ are $2\pi$-periodic in $\phi$, continuously differentiable and $P_i = o(||(r, z)||)$, $Q_i = o(||(r, z)||)$, $i = \overline{1, p}$.

From the construction, we see that system (7) is a differential equation with discontinuous right-hand side. For our needs, we redefine the functions $\tilde{G}$ and $\tilde{g}$ in the neighborhoods of the planes $\mathcal{P}_i$, which contain the surface $\mathcal{S}_i$. In other words, we construct new functions $G_N$ and $g_N$ which are continuous everywhere except possibly at the points $(r, \phi, z) \in \mathcal{P}_i$. The redefinition will be made exceptionally at the points which lie between $\mathcal{P}_i$ and $\mathcal{S}_i$ and belong to the regions $D_i$ or $D_{i+1}$ for each $i$. Therefore, this construction is performed with minimal possible changes corresponding to the $B-$equivalence method [1], which is the main instrument of our investigation.

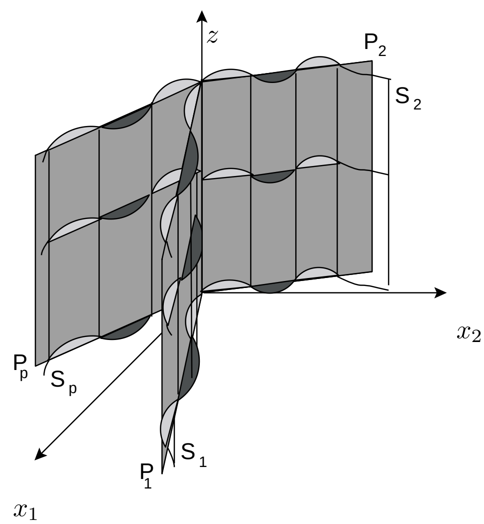


Figure 2: Surfaces $\mathcal{S}_i$, $i = \overline{1, p}$, of discontinuities for the perturbed system (1).

It is clear from the context that if $i = p$ then $D_{p+1} = D_1$. Using the argument above, we realize the following reconstruction of the domain. We consider the subregions of $D_i$ and $D_{i+1}$, which are placed between the plane $\mathcal{P}_i$ and the surface $\mathcal{S}_i$. We refer to the subregions $D_i \cap \tilde{D}_{i+1}$ (light coloured closed regions in Fig. 2) and $D_{i+1} \cap \tilde{D}_i$ (dark coloured closed regions in Fig. 2) for all $i$. We extend the functions $\tilde{G}$ and $\tilde{g}$ from the region $D_i \cap \tilde{D}_{i+1}$ to $D_i$ and from $D_{i+1} \cap \tilde{D}_i$ to $D_{i+1}$ so that the new functions $G_N$ and $g_N$ and their partial derivatives become continuous up to the angle $\phi = \gamma_i$, $i = \overline{1, p}$. According to all these discussions for the definitions of $G_N$ and $g_N$, we conclude that $G_N(r, \phi, z) = \lambda_i r + P_i(r, \phi, z)$ and $g_N(r, \phi, z) = k_i z + Q_i(r, \phi, z)$ for $(r, \phi, z) \in D_i$. Now, we consider the following differential equation

$$\begin{aligned} \frac{dr}{d\phi} &= G_N(r, \phi, z), \\ \frac{dz}{d\phi} &= g_N(r, \phi, z). \end{aligned} \tag{8}$$

Let us fix $i \in \{1, 2, \ldots, p\}$ and consider a neighborhood of $\mathcal{P}_i$ based on the description above. We shall investigate the following three cases:

**I**. Assume that the point $(r, \gamma_i, z) \in \tilde{D}_{i+1}$. Let $(r^0(\phi), (z^0(\phi))$ be a solution of (7) satisfying $(r^0(\gamma_i), (z^0(\gamma_i)) = (\rho, z)$ and $\xi_i$ be the angle where this solution crosses the surface $\mathcal{S}_i$. We denote a solution of (8) on the interval $[\xi_i, \gamma_i]$ by $(r^1(\phi), z^1(\phi))$ with $(r^1(\xi_i), z^1(\xi_i)) = (r^0(\xi_i), z^0(\xi_i))$. Then

$$r^0(\phi) = \exp(\lambda_{i+1}(\phi - \gamma_i))\rho + \int_{\gamma_i}^{\phi} \exp(\lambda_{i+1}(\phi - s)) P_{i+1}(r^0(s), s, z^0(s)) ds,$$

$$z^0(\phi) = \exp(k_{i+1}(\phi - \gamma_i))z + \int_{\gamma_i}^{\phi} \exp(k_{i+1}(\phi - s)) Q_{i+1}(r^0(s), s, z^0(s)) ds,$$

and

$$r^1(\phi) = \exp(\lambda_i(\phi - \xi_i)) r^0(\xi_i) + \int_{\xi_i}^{\phi} \exp(\lambda_i(\phi - s)) P_i(r^1(s), s, z^1(s)) ds.$$

$$z^1(\phi) = \exp(k_i(\phi - \xi_i)) r^0(\xi_i) + \int_{\xi_i}^{\phi} \exp(k_i(\phi - s)) Q_i(r^1(s), s, z^1(s)) ds.$$

Define a mapping $W_i = (W_i^1, W_i^2)$ on the plane $\phi = \gamma_i$ into itself as follows

$$\begin{aligned} W_i^1(\rho, z) &= r^1(\gamma_i) - \rho = [\exp((\lambda_i - \lambda_{i+1})(\gamma_i - \xi_i)) - 1]\rho \\ &+ \exp(\lambda_i(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_{i+1}(\xi_i - s)) P_{i+1} ds \\ &+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_i(\gamma_i - s)) P_i ds, \end{aligned}$$

$$\begin{aligned} W_i^2(\rho,z) &= z^1(\gamma_i) - z = [\exp((k_i - k_{i+1})(\gamma_i - \xi_i)) - 1]z \\ &+ \exp(k_i(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(k_{i+1}(\xi_i - s)) Q_{i+1} ds \\ &+ \int_{\xi_i}^{\gamma_i} \exp(k_i(\gamma_i - s)) Q_i ds. \end{aligned}$$

**II**. If the point $(r,\gamma_i,z) \in \tilde{D}_i$, we can evaluate $W_i$ in the same way:

$$\begin{aligned} W_i^1(\rho,z) &= [\exp((\lambda_i - \lambda_{i+1})(\xi_i - \gamma_i)) - 1]\rho \\ &+ \exp(\lambda_{i+1}(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_i(\xi_i - s)) P_i ds \\ &+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_{i+1}(\gamma_i - s)) P_{i+1} ds, \end{aligned}$$

$$\begin{aligned} W_i^2(\rho,z) &= [\exp((k_i - k_{i+1})(\xi_i - \gamma_i)) - 1]z \\ &+ \exp(k_{i+1}(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(k_i(\xi_i - s)) Q_i ds \\ &+ \int_{\xi_i}^{\gamma_i} \exp(\lambda_{i+1}(\gamma_i - s)) Q_{i+1} ds. \end{aligned}$$

**III**. If $(r,\gamma_i,z) \in \mathcal{S}_i$, then $W_i(\rho,z) = 0$.

Results from [2] imply that the functions $W_i^1$ and $W_i^2$, $i = \overline{1,p}$, are continuously differentiable and we have $W_i^1 = o(||(\rho,z)||)$, $W_i^2 = o(||(\rho,z)||)$, which follows from the equation (5). In addition, we note that there exists a Lipschitz constant $\ell$ and a bounded function $m(\ell)$ [1, 2] such that

$$\|W_i^j(\rho_1,z_1) - W_i^j(\rho_2,z_2)\| \le \ell m(\ell)(\|\rho_1 - \rho_2\| + \|z_1 - z_2\|), \tag{9}$$

for all $\rho_1,\rho_2,z_1,z_2 \in \mathbb{R}, j = 1,2.$

Let $(r(\phi,r_0), z(\phi,z_0))$ be a solution of (7) with $r(0,r_0) = r_0$, $z(0,z_0) = z_0$ and $\xi_i$ be the meeting angle of this solution with the surface $\mathcal{S}_i$, $i = \overline{1,p}$. Denote by $(\xi_i, .\gamma_i]$ the interval $(\xi_i,\gamma_i]$ whenever $\xi_i \le \gamma_i$ and $[\gamma_i,\xi_i)$ if $\gamma_i < \xi_i$. For sufficiently small $\Omega$, the solution $r(\phi,r_0)$, whose trajectory is in $\Omega$ for all $\phi \in [0,2\pi]$, takes the same values with the exception of the oriented intervals $(\xi_i,.\gamma_i]$ as the solution $(\rho(\phi,r_0), z(\phi,z_0))$ with $\rho(0,r_0) = r_0$, $z(0,z_0) = z_0$ of the impulsive differential equation

$$\begin{aligned} &\frac{d\rho}{d\phi} = G_N(\rho,\phi,z), \\ &\frac{dz}{d\phi} = g_N(\rho,\phi,z), \qquad \phi \ne \gamma_i, \\ &\Delta\rho|_{\phi = \gamma_i} = W_i^1(\rho,z), \\ &\Delta z|_{\phi = \gamma_i} = W_i^2(\rho,z). \end{aligned} \tag{10}$$

That is, systems (7) and (10) are said to be $B-$equivalent in the sense of the definition in [2]. From the discussion and the construction above, it implies that solutions of (7) exist in the neighborhood $\Omega$, they are continuous and have discontinuities in the derivative on the surface $\mathcal{S}_i$ for each $i$. Accordingly, a solution of system (6) starting at any initial point is continuous, continuously differentiable except possibly at the moments when the trajectories intersect the surface $\mathcal{S}_i$ and is unique.

# 4 Center manifold

We establish a center manifold theorem for sufficiently small solutions to (10), that is, we show that these solutions can be captured on a 2-dimensional invariant manifold and we explicitly describe the dynamics on this manifold.

The functions $G_N$ and $g_N$ in (10) have been defined as $G_N(r,\phi,z) = \lambda_i r + P_i(r,\phi,z)$ and $g_N(r,\phi,z) = k_i z + Q_i(r,\phi,z)$, where $(r,\phi,z) \in D_i$. Functions $P_i$ and $Q_i$ are $2\pi-$periodic in $\phi$, and satisfy in a neighborhood of the origin

$$\|P_i(\rho,\phi,z) - P_i(\rho',\phi,z')\| \le L(\|\rho-\rho'\| + \|z-z'\|), \tag{11}$$
$$\|Q_i(\rho,\phi,z) - Q_i(\rho',\phi,z')\| \le L(\|\rho-\rho'\| + \|z-z'\|), \tag{12}$$

for sufficiently small positive constant $L$, $i=\overline{1,p}$. Applying the methods of the paper [6], we can conclude that system (10) has two integral manifolds whose equations are given by:

$$\begin{aligned}\Phi_0(\phi,\rho) &= \int_{-\infty}^{\phi} e^{k(\phi-s)} Q(\rho(s,\phi,\rho),s,z(s,\phi,\rho))ds \\ &\quad + \sum_{\gamma_i<\phi} e^{k_i(\phi-\gamma_i)} W_i^2(\rho(\gamma_i,\phi,\rho),z(\gamma_i,\phi,\rho)),\end{aligned} \tag{13}$$

and

$$\begin{aligned}\Phi_-(\phi,z) &= -\int_{\phi}^{\infty} e^{\lambda(\phi-s)} P(\rho(s,\phi,z),s,z(s,\phi,z))ds \\ &\quad + \sum_{\gamma_i<\phi} e^{\lambda_i(\phi-\gamma_i)} W_i^1(\rho(\gamma_i,\phi,z),z(\gamma_i,\phi,z)),\end{aligned} \tag{14}$$

where $k=k_i, \lambda=\lambda_i, P=P_i$ and $Q=Q_i$ whenever $(s,\cdot,\cdot)\in D_i$. The pair $(\rho(s,\phi,\rho),z(s,\phi,\rho))$ in (13) denotes a solution of (10) satisfying $\rho(\phi,\phi,\rho)=\rho$ and $(\rho(s,\phi,z),z(s,\phi,z))$, in (14), is a solution of (10) with $z(\phi,\phi,z)=z$.

It is also shown in [6] that there exist positive constants $K_0, M_0, \sigma_0$ such that $\Phi_0$ satisfies:

$$\Phi_0(\phi,0)=0, \tag{15}$$
$$\|\Phi_0(\phi,\rho_1) - \Phi_0(\phi,\rho_2)\| \le K_0\ell\|\rho_1-\rho_2\|, \tag{16}$$

for all $\rho_1, \rho_2$ such that a solution $w(\phi) = (\rho(\phi), z(\phi))$ of (10) with $w(\phi_0) = (\rho_0, \Phi_0(\phi_0, \rho_0)), \rho_0 \geq 0$, is defined on $\mathbb{R}$ and has the following property:

$$\|w(\phi)\| \leq M_0 \rho_0 e^{-\sigma_0(\phi-\phi_0)}, \quad \phi \geq \phi_0. \tag{17}$$

Furthermore, it is shown that there exist positive constants $K_-, M_-, \sigma_-$ such that $\Phi_-$ satisfies:

$$\Phi_-(\phi, 0) = 0, \tag{18}$$

$$\|\Phi_-(\phi, z_1) - \Phi_-(\phi, z_2)\| \leq K_- \ell \|z_1 - z_2\|, \tag{19}$$

for all $z_1, z_2$ such that a solution $w(\phi) = (\rho(\phi), z(\phi))$ of (10) with $w(\phi_0) = (\Phi_-(\phi_0, z_0), z_0), z_0 \in \mathbb{R}$, is defined on $\mathbb{R}$ and satisfies

$$\|w(\phi)\| \leq M_- \|z_0\| e^{-\sigma_-(\phi-\phi_0)}, \quad \phi \leq \phi_0. \tag{20}$$

Denote $S_0 = \{(\rho, \phi, z) : z = \Phi_0(\phi, \rho)\}$ and $S_- = \{(\rho, \phi, z) : \rho = \Phi_-(\phi, z)\}$. Here, $S_0$ is said to be the *center manifold* and $S_-$ is said to be the *stable manifold*.

The following lemmas can be proven in a similar manner to the ones in [6] with slight changes.

LEMMA 4.1 *If the Lipschitz constant $\ell$ is sufficiently small, then for every solution $w(\phi) = (\rho(\phi), z(\phi))$ of (10) there exists a solution $\mu(\phi) = (u(\phi), v(\phi))$ on the center manifold, $S_0$, such that*

$$\begin{aligned} &\|\rho(\phi) - u(\phi)\| \leq 2M_0 \|\rho(\phi_0) - u(\phi_0)\| e^{-\sigma_0(\phi-\phi_0)}, \\ &\|z(\phi) - v(\phi)\| \leq M_0 \|z(\phi_0) - v(\phi_0)\| e^{-\sigma_0(\phi-\phi_0)}, \quad \phi \geq \phi_0, \end{aligned} \tag{21}$$

*where $M_0$ and $\sigma_0$ are the constants used in (17).*

LEMMA 4.2 *For sufficiently small Lipschitz constant $\ell$, the surface $S_0$ is stable in large.*

The dynamics reduced to local the center manifold $S_0$ is governed by an impulsive differential equation that is satisfied by the first coordinate of the solutions of (10) and has the form:

$$\begin{aligned} &\frac{d\rho}{d\phi} = G_N(\rho, \phi, \Phi_0(\phi, \rho)), \quad \phi \neq \gamma_i, \\ &\Delta\rho|_{\phi=\gamma_i} = W_i^1(\rho, \Phi_0(\phi, \rho)). \end{aligned} \tag{22}$$

The following theorem follows from the reduction principle.

THEOREM 4.1 *Assume that the conditions assumed so far are fulfilled. Then the trivial solution of (10) is stable, asymptotically stable or unstable if the trivial solution of (22) is stable, asymptotically stable or unstable, respectively.*

Using $B-$equivalence, one can see that the following theorem holds:

THEOREM 4.2 *Assume that the conditions given above are fulfilled. Then the trivial solution of (6) is stable, asymptotically stable or unstable if the trivial solution of (22) is stable, asymptotically stable or unstable, respectively.*

# 5 Hopf bifurcation

The center manifold reduction in the previous section allows us to establish a Hopf bifurcation theorem, yielding a very powerful tool to perform a bifurcation analysis on parameter dependent versions of the considered systems. During the last two decades, many authors have contributed towards developing the general theory.

In order to state the Hopf bifurcation theorem, we include parameter dependence into our framework. In particular, the bifurcation of periodic solutions under the influence of a single parameter $\mu$, $\mu \in (-\mu_0, \mu_0)$, $\mu_0$ a positive constant, is considered for the system:

$$\begin{aligned} \frac{dx}{dt} &= \hat{F}(x, z, \mu), \\ \frac{dz}{dt} &= \hat{f}(x, z, \mu), \end{aligned} \tag{23}$$

where $\hat{F}(x, z, \mu) = A_i x + f_i(x, z) + \mu F_i(x, z, \mu)$ and $\hat{f}(x, z, \mu) = b_i z + h_i(x, z) + \mu H_i(x, z, \mu)$ whenever $(x, z) \in \tilde{D}_i(\mu) \subset \mathbb{R}^3$, which will be defined below. We will need the following assumptions on the system (23):

(H1) Let $\{\mathcal{S}_i(\mu)\}_{i=1}^p$ be a collection of surfaces in $\Omega$ which start at the $z$-axis, i.e., $\mathcal{S}_i(\mu) = c_i(\mu) \times \mathbb{R}$, where $c_i(\mu)$ are curves given by $\langle a^i, x\rangle + \tau_i(x) + \mu\kappa_i(x, \mu) = 0$, $x \in \mathbb{R}^2$, $i = \overline{1, p}$;

(H2) Let $\{\mathcal{P}_i(\mu)\}_{i=1}^p$ be a union of half-planes which start at the the $z-$axis, i.e., $\mathcal{P}_i(\mu) = l_i(\mu) \times \mathbb{R}$, where $l_i(\mu)$ is defined by $\langle a^i + \mu \dfrac{\partial \kappa_i(0, \mu)}{\partial x}, x\rangle = 0$, $i = \overline{1, p}$. Denote by $\gamma_i(\mu)$ the angle of the line $l_i(\mu)$, $i = 1, 2, \ldots, p$.

Like the construction of the regions $D_i$ and $\tilde{D}_i$, we define for $\mu \in (-\mu_0, \mu_0)$, $i = 2, 3, \ldots, p$, the ones associated to the system (23):

$$\tilde{D}_1(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_p(\mu) + \Psi_p < \phi \leq \gamma_1(\mu) + 2\pi + \Psi_1,\ z \in \mathbb{R}\},$$

$$\tilde{D}_i(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) + \Psi_{i-1} < \phi \leq \gamma_i(\mu) + \Psi_i,\ z \in \mathbb{R}\},$$

$$D_1(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_p(\mu) < \phi \leq \gamma_1(\mu) + 2\pi,\ z \in \mathbb{R}\},$$

$$D_i(\mu) = \{(r, \phi, z, \mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) < \phi \leq \gamma_i(\mu),\ z \in \mathbb{R}\}.$$

Here the functions $\Psi_i = \Psi_i(r, \phi, \mu)$ are $2\pi-$periodic in $\phi$, continuously differentiable, $\Psi_i = O(r)$, $i = \overline{1, p}$ and can defined in a similar manner to $\psi_i$ in (5).

To establish the Hopf bifurcation theorem, we also need the following assumptions:

(H3) The functions $F_i : N_\varepsilon(\tilde{D}_i(\mu)) \to \mathbb{R}^2$ and $\kappa_i$ are analytical functions in $x$, $z$ and $\mu$ in the $\varepsilon-$neighbourhood of their domains;

(H4) $F_i(0,0,\mu)=0$ and $\kappa_i(0,\mu)=0$ hold uniformly for $\mu\in(-\mu_0,\mu_0)$;

(H5) The matrices $A_i$, the constants $b_i$, the functions $f_i$ $g_i$, $\tau_i$ and the constant vectors $a^i$ correspond to the ones described in systems (1) and (6).

In cylindrical coordinates, system (23) reduces to

$$\begin{aligned}\frac{dr}{d\phi}&=\hat{G}(r,\phi,z,\mu),\\ \frac{dz}{d\phi}&=\hat{g}(r,\phi,z,\mu),\end{aligned}\tag{24}$$

$\hat{G}(r,\phi,z,\mu)=\lambda_i(\mu)r+P_i(r,\phi,z,\mu)$ and $\hat{g}(r,\phi,z,\mu)=k_i(\mu)z+Q_i(r,\phi,z,\mu)$ if $(r,\phi,z,\mu)\in\tilde{D}_i(\mu)$.

Let the following impulse system

$$\begin{aligned}&\frac{d\rho}{d\phi}=\hat{G}_N(\rho,\phi,z,\mu),\\ &\frac{dz}{d\phi}=\hat{g}_N(\rho,\phi,z,\mu),\qquad \phi\neq\gamma_i(\mu),\\ &\Delta\rho|_{\phi=\gamma_i(\mu)}=W_i^1(\rho,z,\mu)\\ &\Delta z|_{\phi=\gamma_i(\mu)}=W_i^2(\rho,z,\mu)\end{aligned}\tag{25}$$

be $B-$equivalent to (24), where $\hat{G}_N$ and $\hat{g}_N$ stand, respectively, for the extensions of $\hat{G}$ and $\hat{g}$. That is, $\hat{G}_N(\rho,\phi,z,\mu)=\lambda_i(\mu)\rho+P_i(\rho,\phi,z,\mu)$ and $\hat{g}_N(\rho,\phi,z,\mu)=k_i(\mu)z+Q_i(\rho,\phi,z,\mu)$ for $(\rho,\phi,z,\mu)\in D_i(\mu)$. Then the functions $\hat{G}_N$ and $\hat{g}_N$ and their partial derivatives become continuous up to the angle $\phi=\gamma_i(\mu)$ for $i=\overline{1,p}$. The functions $W_i^1(\rho,z,\mu)$ and $W_i^2(\rho,z,\mu)$ can be defined in the same manner as in Section 3.

Following the same methods which are used to obtain (13) and (14), we can say that system (25) has two integral manifolds whose equations are given by:

$$\begin{aligned}\Phi_0(\phi,\rho,\mu)&=\int_{-\infty}^{\phi}e^{k(\mu)(\phi-s)}Q(\rho(s,\phi,\rho,\mu),s,z(s,\phi,\rho,\mu),\mu)ds\\ &+\sum_{\gamma_i(\mu)<\phi}e^{k_i(\mu)(\phi-\gamma_i(\mu))}W_i^2(\rho(\gamma_i(\mu),\phi,\rho,\mu),z(\gamma_i(\mu),\phi,\rho,\mu),\mu),\end{aligned}\tag{26}$$

and

$$\begin{aligned}\Phi_-(\phi,z,\mu)&=-\int_{\phi}^{\infty}e^{\lambda(\mu)(\phi-s)}P(\rho(s,\phi,z,\mu),s,z(s,\phi,z,\mu),\mu)ds\\ &+\sum_{\gamma_i(\mu)<\phi}e^{\lambda_i(\mu)(\phi-\gamma_i(\mu))}W_i^1(\rho(\gamma_i(\mu),\phi,z,\mu),z(\gamma_i(\mu),\phi,z,\mu),\mu),\end{aligned}\tag{27}$$

where $k(\mu)=k_i(\mu),\lambda(\mu)=\lambda_i(\mu),P=P_i$ and $Q=Q_i$ whenever $(s,\cdot,\cdot,\cdot)\in D_i(\mu)$. In (26), the pair $(\rho(s,\phi,\rho,\mu),z(s,\phi,\rho,\mu))$ denotes a solution of (25) satisfying $\rho(\phi, \phi, \rho, \mu) = \rho$. Similarly, $(\rho(s, \phi, z, \mu), z(s, \phi, z, \mu))$, in (27), is a solution of (25) with $z(\phi, \phi, z, \mu) = z$.

Set $S_0(\mu) = \{(\rho, \phi, z, \mu) : z = \Phi_0(\phi, \rho, \mu)\}$ and $S_-(\mu) = \{(\rho, \phi, z, \mu) : \rho = \Phi_-(\phi, z, \mu)\}$.

The reduced system on the center manifold $S_0(\mu)$ is given by

$$\begin{aligned} &\frac{d\rho}{d\phi} = \hat{G}_N(\rho, \phi, \Phi_0(\phi, \rho, \mu), \mu), \quad \phi \neq \gamma_i(\mu), \\ &\Delta\rho\mid_{\phi=\phi_i(\mu)} = W_i^1(\rho, \Phi_0(\phi, \rho, \mu), \mu). \end{aligned} \tag{28}$$

Similar to (3) and (4) we can define the functions

$$q_1(\mu) = \exp(\sum_{i=1}^{p} \lambda_i(\mu)\theta_i(\mu)), \tag{29}$$

$$q_2(\mu) = \exp(\sum_{i=1}^{p} k_i(\mu)\theta_i(\mu)). \tag{30}$$

System (28) is a system of the type studied in [2] and there it is shown that this system, for sufficiently small $\mu$, has a periodic solution with period $2\pi$. For our needs, we shall show that if the first coordinate of a solution of (25) is $2\pi-$periodic, then so is the second one.

Now, since

$$\rho(s + 2\pi, \phi + 2\pi, \rho, \mu) = \rho(s, \phi, \rho, \mu),$$
$$z(s + 2\pi, \phi + 2\pi, \rho, \mu) = z(s, \phi, \rho, \mu),$$

and each $Q_i$ is $2\pi-$periodic in $\phi$, we have

$$\begin{aligned} &\Phi_0(\phi + 2\pi, \rho, \mu) \\ &= \int_{-\infty}^{\phi+2\pi} e^{k(\mu)(\phi+2\pi-s)} Q(\rho(s, \phi + 2\pi, \rho, \mu), s, z(s, \phi + 2\pi, \rho, \mu), \mu) ds \\ &\quad + \sum_{\gamma_i(\mu)<\phi+2\pi} e^{k_i(\mu)(\phi+2\pi-\gamma_i(\mu))} \times \\ &\qquad \times\ W_i^2(\rho(\gamma_i(\mu), \phi + 2\pi, \rho, \mu), z(\gamma_i(\mu), \phi + 2\pi, \rho, \mu), \mu) \\ &= \int_{-\infty}^{\phi} e^{k(\mu)(\phi-t)} Q(\rho(t, \phi, \rho, \mu), t, z(t, \phi, \rho, \mu), \mu) dt \\ &\quad + \sum_{\bar{\gamma}_i(\mu)<\phi} e^{k_i(\mu)(\phi-\bar{\gamma}_i(\mu))} W_i^2(\rho(\bar{\gamma}_i(\mu), \phi, \rho, \mu), z(\bar{\gamma}_i(\mu), \phi, \rho, \mu), \mu) \\ &= \Phi_0(\phi, \rho, \mu), \end{aligned}$$

where the substitutions $s = t + 2\pi$ and $\gamma_i(\mu) = \bar{\gamma}_i(\mu) + 2\pi$ are used for the integral and summation in the second equality.

Then, we obtain the following theorem whose proof for two dimensional case can be found in [2] .

THEOREM 5.1 *Assume that* $q_1(0) = 1, q_1'(0) \neq 0, q_2(0) < 1$, *and the origin is a focus for (6). Then, for sufficiently small* $r_0$ *and* $z_0$, *there exists a unique continuous function* $\mu = \delta(r_0, z_0)$, $\delta(0,0) = 0$ *such that the solution* $(r(\phi, \delta(r_0, z_0)), z(\phi, \delta(r_0, z_0)))$ *of (24), with the initial condition* $(r(0, \delta(r_0, z_0)), z(0, \delta(r_0, z_0)) = (r_0, z_0)$, *is periodic with period* $2\pi$. *The period of the corresponding periodic solution of (23) is* $\sum_{i=1}^{p} \frac{\theta_i}{\beta_i} + o(|\mu|)$.

# 6 An example

For convenience in this section, we shall use the corresponding notations that are adopted above.

EXAMPLE 6.1 *Let* $c_1(\mu)$ *and* $c_2(\mu)$ *denote the curves determined by* $x_2 = \frac{1}{\sqrt{3}}x_1 + (1+\mu)x_1^3$, $x_1 > 0$ *and* $x_2 = \sqrt{3}x_1 + x_1^5 + \mu x_1^2$, $x_1 < 0$ , *respectively. We choose*

$$A_1 = \begin{bmatrix} -0.7 & -2 \\ 2 & -0.7 \end{bmatrix}, \ f_1(x,z) = \begin{bmatrix} x_1 z\sqrt{x_1^2+x_2^2} \\ x_2 z^2\sqrt{x_1^2+x_2^2} \end{bmatrix}, \ F_1(x,z,\mu) = \begin{bmatrix} x_1(1+z) \\ x_2 \end{bmatrix},$$

$$b_1 = 2, \ h_1(x,z) = x_1^2 z, \ H_1(x,z,\mu) = z,$$

$$A_2 = \begin{bmatrix} 0.5 & -2 \\ 2 & 0.5 \end{bmatrix}, \ f_2(x,z) = \begin{bmatrix} -2x_1 z^2\sqrt{x_1^2+x_2^2} \\ -2x_2\sqrt{x_1^2+x_2^2} \end{bmatrix}, \ F_2(x,z,\mu) = \begin{bmatrix} x_1 \\ x_2(1+x_1 z) \end{bmatrix},$$

$$b_2 = -1.5, \ h_2(x,z) = x_1 z, \ H_2(x,z,\mu) = [1-(x_1^2+x_2^2)]z.$$

*After these preparations, we consider the system*

$$\begin{aligned} \frac{dx}{dt} &= \hat{F}(x,z,\mu), \\ \frac{dz}{dt} &= \hat{f}(x,z,\mu), \end{aligned} \tag{31}$$

*where* $\hat{F}(x,z,\mu) = A_i x + f_i(x,z) + \mu F_i(x,z,\mu)$ *and* $\hat{f}(x,z,\mu) = b_i z + h_i(x,z) + \mu H_i(x,z,\mu)$ *whenever* $(x,z) \in \tilde{D}_i(\mu)$.

*Since* $l_1(\mu)$ $(l_2(\mu))$ *coincides with* $l_1$ $(l_2)$, $\gamma_1 = \gamma_1(\mu) = \frac{\pi}{6}$ *and* $\gamma_2 = \gamma_2(\mu) = \frac{4\pi}{3}$. *Now, we can evaluate* $q_1(\mu)$ *and* $q_2(\mu)$ *as follows:*

$$q_1(\mu) = \exp(\pi\mu), \tag{32}$$

$$q_2(\mu) = \exp(\pi(\mu - \frac{1}{24})). \tag{33}$$

*From (32) and (33), we can see that* $q_1(0) = 1$, $q_1'(0) > 0$ *and* $q_2(0) < 1$. *Therefore, by Theorem 5.1, system (31) has a periodic solution with period* $\approx \pi$. *One can see from the Figures* 3 *and* 4 *below, which are obtained for different initial conditions, that the trajectories approach to the periodic solution from above and below. In other words, system* (31) *admits a limit cycle.*

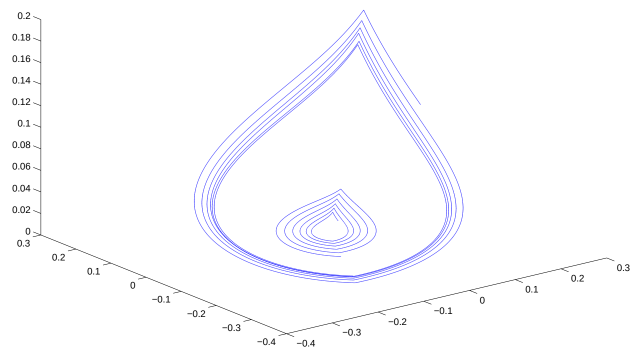


Figure 3: Existence of a periodic solution for (31)

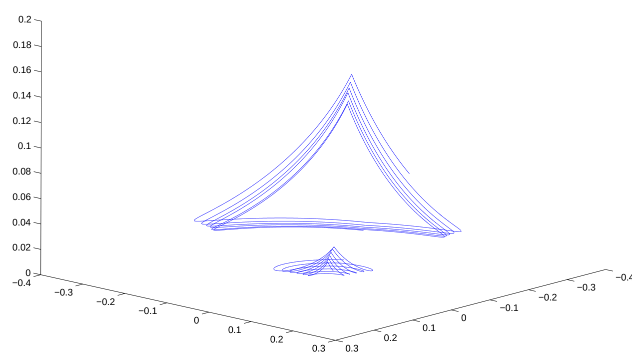


Figure 4: Existence of another periodic solution for (31) with a different initial value.

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