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Bifurcation of a non-smooth planar limit cycle from a vertex[☆]

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ABSTRACT

We investigate non-smooth planar systems of differential equations with discontinuous right-hand sides. Discontinuity sets intersect at a vertex, and are of quasilinear nature. By means of the *B*-equivalence method, which was introduced in [M. Akhmetov, Asymptotic representation of solutions of regularly perturbed systems of differential equations with a nonclassical right-hand side, Ukrainian Math. J. 43 (1991) 1209–1214; M. Akhmetov, On the expansion of solutions to differential equations with discontinuous right-hand side in a series in initial data and parameters, Ukrainian Math. J. 45 (1993) 786–789; M. Akhmetov, N.A. Perestyuk, Differential properties of solutions and integral surfaces of nonlinear impulse systems, Differential Equations 28 (1992) 445–453] (see also [E. Akalın, M.U. Akhmet, The principles of *B*-smooth discontinuous flows, Math. Comput. Simul. 49 (2005) 981–995; M.U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, Nonlinear Anal. 60 (2005) 163–178]), these systems are reduced to impulsive differential equations. Sufficient conditions are established for the existence of foci and centers both in the noncritical and critical cases. Hopf bifurcation is considered from a vertex, which unites several curves, in the critical case. An appropriate example is provided to illustrate the results.

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1. Introduction

The theory of differential equations with discontinuous right-hand sides has been substantially developed through numerous applications. There are many problems from mechanics, engineering sciences [1–4], control theory [5] and economics [6] that are modeled by dynamical systems with discontinuous vector fields. Besides, the books [1,4,7], which concern mechanical systems with dry friction, periodic solutions of discontinuous systems and discontinuous oscillations, form an important basis for the development of such discontinuous systems. Owing to the problems of applied nature, a qualitative theory of classical ordinary differential equations including the notions of existence, uniqueness, continuous dependence, stability and bifurcation has been carefully adapted for equations with discontinuous right-hand sides. The main trends of the theory can be found in [5].

Bifurcations in smooth systems are well understood [8–11], but little is known in discontinuous systems. Stimulated by non-smooth phenomena in the real world, the subject of Hopf bifurcation in discontinuous systems has received great attention in recent years [2–4,12–18]. Dankowicz and Nordmark [19] study bifurcations of stick-slip oscillations for the friction model which leads to a non-smooth dynamical system having discontinuity at the first derivative of the vector field. Feigin [20,21] considers *C*-bifurcations, also known as border-collision bifurcations, in Filippov systems which are

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a subclass of discontinuous systems described by differential equations with a discontinuous right-hand side [5]. Border-collision bifurcations for non-smooth discrete maps are also addressed by Nusse and Yorke [15,16].

Kunze [3] and Küpper et al. [13,18] address bifurcation of periodic solutions for planar Filippov systems with discontinuities on a single straight line. In [18] generalized Hopf bifurcation for the following piecewise smooth planar system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{cases} f^{+}(x, y, \lambda), & x > 0, \\ f^{-}(x, y, \lambda), & x < 0, \end{cases}$$

where $f^{\pm}(x, y, \lambda) = A^{\pm}(\lambda)(x, y)^{\mathrm{T}} + g^{\pm}(x, y, \lambda)$, λ a real parameter, has been investigated using differential inclusions. Eigenvalues of the matrix $A^{\pm}(\lambda)$ were assumed to be complex conjugate, i.e., $\alpha^{\pm}(\lambda) \pm \mathrm{i}\omega^{\pm}(\lambda)$. This system has been stimulated by a brake system of the form

$$\begin{aligned} &mu'' + d_1u' + c_1u = \sigma^{+}(u, u', \lambda), \quad \text{if } u > 0, \\ &mu'' + (d_1 + d_2)u' + (c_1 + c_2)u = \sigma^{-}(u, u', \lambda), \quad \text{if } u < 0, \end{aligned}$$

where a mass m rests on a smooth surface and is connected to the walls by springs (c_1 and c_2) and dampers (d_1 and d_2). $\sigma^{\pm}$ denotes the external force and the parameter λ controls its magnitude (see [18] for details).

In papers [17,22], possibly for the first time, a special structure of the domain has been developed for planar differential equations with discontinuities. To say more clearly, [22] treats bifurcation of periodic solutions for planar discontinuous dynamical systems where discontinuities in the state variable appear on countably many curves intersecting at the origin, and [17] studies generalized Hopf bifurcation for piecewise smooth planar systems with discontinuities on several straight lines emanating from the origin. We suppose that domains of this type can be useful in mechanical and electrical models with discontinuities under proper transformations.

Most of the papers in the literature assume that discontinuity sets of non-smooth systems consist of a single surface, especially a straight line [3,12,13,18]. However, due to exterior effects, discontinuities may appear on curves or surfaces of nonlinear feature. Hence, it is reasonable to perturb the sets of discontinuities. Differential equations whose right-hand sides are discontinuous on nonlinear surfaces were investigated in [23,24] by the method of B-equivalence. This method was first proposed to reduce impulsive systems with variable time of impulses to the systems with fixed moments of impulse effects. It then turned out that this method could be used for differential equations with discontinuous right-hand side as well [23,25]. That is, through the B-equivalence method, differential equations with discontinuous vector fields with nonlinear discontinuity sets can be reduced to impulsive differential equations with fixed moments of impulses.

Our present work is an attempt to generalize the problem of Hopf bifurcation for a planar non-smooth system by considering discontinuities on finitely many nonlinear curves emanating from a vertex. We consider the domain in a neighborhood of a vertex which unites several curves. That is, the phase space is divided into subdomains and the system is described by a different set of differential equations in each domain. We can say that the system considered in this paper is more general than the one in [17], where discontinuities occur at straight lines. We aim to give some theoretical background rather than applications, which will be very useful in many problems in the future. Using B-equivalence [22–26] of the issue systems to impulsive differential equations, we obtain corresponding qualitative properties. It is the inherent advantage of the B-equivalence method that we can study equations with nonlinear discontinuity sets.

The paper is organized in the following way. In Section 2, we introduce the nonperturbed system and study existence of foci and centers for that system. Section 3 presents the perturbed system and the notion of B-equivalent impulsive systems. The problem of distinguishing between the center and the focus is solved in Section 4. We investigate bifurcation of periodic solutions in the next section. Afterwards, an appropriate example is worked out to illustrate our results. Finally, we discuss the possible generalization of the present results in Section 7.

2. The nonperturbed system

Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively. Let $\mathbb{R}^2$ be the 2-dimensional real space and $\langle x, y\rangle$ denote the scalar product for all vectors $x, y \in \mathbb{R}^2$. The norm of a vector $x \in \mathbb{R}^2$ is given by $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$.

For the sake of brevity in the sequel, every angle for a point is considered with respect to the positive half-line of the first coordinate axis.

In the rest of the present paper, following assumptions will be needed.

(A1) Let $\{l_i\}_{i=1}^{p}$, $p \geq 2$, $p \in \mathbb{N}$, be a set of half-lines starting at the origin and given by the equations $\Phi_i(x) = 0$, $\Phi_i(x) = \langle a^i, x\rangle$, $i = 1, 2, \ldots, p$, where $a^i = ({a_1}^i, {a_2}^i) \in \mathbb{R}^2$ are constant vectors (see Fig. 1). Let γ_i, $i = 1, 2, \ldots, p$, denote the angles of the lines l_i such that

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_p < 2\pi\,.$$

(A2) There exist real-valued constant 2×2 matrices $A_1, A_2, \ldots, A_p$ defined by $A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}$ with $\beta_i > 0$ for each $i = 1, 2, \ldots, p$.

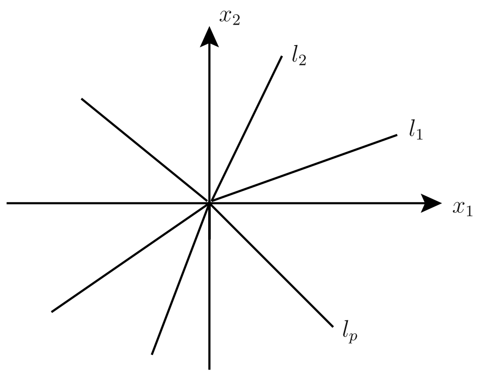

Fig. 1. The domain of the nonperturbed system (1) with a vertex which unites the straight lines l_i, $i = 1, 2, \ldots, p$.

Meanwhile, for convenience throughout this paper, we adopt the notations below.

(N1) $\theta_1 = (2\pi + \gamma_1) - \gamma_p, \theta_i = \gamma_i - \gamma_{i-1}, i = 2, 3, \ldots, p.$

(N2) Let D_i denote the region situated between the straight lines l_{i-1} and l_i and defined in polar coordinates (r, ϕ), where $x_1 = r\cos\phi,\ x_2 = r\sin\phi$, as follows

$$D_1 = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_p < \phi \leq \gamma_1 + 2\pi\},$$
$$D_i = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_{i-1} < \phi \leq \gamma_i\}, \quad i = 2, 3, \ldots, p.$$

Now we define a function f such that $f(x) = A_i x$ for $x \in D_i$, $i = 1, 2, \ldots, p$, and consider a differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x). \tag{1}$$

According to the definition of the regions D_i, one can see that the function f in system (1) has discontinuities on the straight lines l_i, $i = 1, 2, \ldots, p$.

Remark 2.1. It follows from the assumptions (A1) and (A2) that

$$\left\langle \frac{\partial \Phi_i(x)}{\partial x}, f(x) \right\rangle \neq 0 \quad \text{for } x \in l_i,\ i = 1, 2, \ldots, p.$$

That is, the vector field is transversal at every point on l_i for each i.

Using the polar transformation, we can write (1) in the following form

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = g(r), \tag{2}$$

where

$$g(r) = \begin{cases} \lambda_1 r, & \text{if } \phi \in (\gamma_p + 2k\pi, \gamma_1 + 2(k+1)\pi], \\ \lambda_i r, & \text{if } \phi \in (\gamma_{i-1} + 2k\pi, \gamma_i + 2k\pi], \quad i = 2, 3, \ldots, p, \end{cases}$$

with $\lambda_i = \frac{\alpha_i}{\beta_i}, i = 1, 2, \ldots, p$, and $k \in \mathbb{Z}$. Since Eq. (2) is 2π-periodic, it will be enough to consider just the section $\phi \in [0, 2\pi]$. Thus, the function g in (2) can be defined shortly as $g(r) = \lambda_i r$ if $(r, \phi) \in D_i$. Clearly, this function has discontinuities when $\phi = \gamma_i, i = 1, 2, \ldots, p$.

The solution $r(\phi, r_0)$ of (2) starting at the point $(0, r_0)$ has the form

$$r(\phi, r_0) = \begin{cases} \exp(\lambda_1 \phi) r_0, & \text{if } 0 \leq \phi \leq \gamma_1, \\ \exp\left(\lambda_1\gamma_1 + \lambda_2\theta_2 + \cdots + \lambda_i(\phi - \gamma_{i-1})\right) r_0, & \text{if } \gamma_{i-1} < \phi \leq \gamma_i, \\ \exp\left(\lambda_1\left(\phi - (\gamma_p - \gamma_1)\right) + \displaystyle\sum_{i=2}^{p} \lambda_i\theta_i\right) r_0, & \text{if } \gamma_p < \phi \leq 2\pi, \end{cases}$$

where $i = 2, 3, \ldots, p$.

If we construct the Poincaré return map $r(2\pi, r_0)$ on the positive half-axis Ox_1, we get $r(2\pi, r_0) = \exp(\sum_{i=1}^{p} \lambda_i\theta_i) r_0$.

It is well-known that the origin is said to be a *center* if there exists a neighborhood of the origin where all trajectories are cycles surrounding the origin. Besides, if we can find a neighborhood of the origin such that all trajectories starting in it spiral to the origin as $t \to \infty$ ($t \to -\infty$), we call the origin as a stable (unstable) *focus*.

Let us denote $q = \exp(\sum_{i=1}^{p} \lambda_i\theta_i)$. Since $r(2\pi, r_0) = qr_0$, we obtain the following theorem for the nonperturbed system.

Theorem 2.1. *If*

(i) $q = 1$, *then the origin is a center and all solutions are periodic with period* $T = \sum_{i=1}^{p} \frac{\theta_i}{\beta_i}$;
(ii) $q < 1$, *then the origin is a stable focus;*
(iii) $q > 1$, *then the origin is an unstable focus of* (1).

3. The perturbed system

Let $\Omega \subset \mathbb{R}^2$ be a domain in the neighborhood of the origin. The following is the list of conditions assumed for this section. In what follows we use notations $O(s)$, $o(s)$ assuming that $s > 0, s \to 0$.

(P1) Let $\{c_i\}_{i=1}^{p}$ be a set of curves in Ω which start at the origin and are determined by the equations $\tilde{\Phi}_i(y) = 0$, $\tilde{\Phi}_i(y) = \langle a^i, y\rangle + \tau_i(y)$, $i = 1, 2, \ldots, p$, where $\tau_i(y) = o(\|y\|)$ and for each i, the constant vectors a^i are the same as described in (A1).

We split the domain Ω into p-subdomains, which will be called $\tilde{D}_i$ and formulated soon, by means of the curves c_i, $i = 1, 2, \ldots, p$. We assume without loss of generality that $\gamma_i \neq \frac{\pi}{2} j$, $j = 1, 3$. Then for sufficiently small r, equation of the curve c_i can be written in polar coordinates as follows [22]

$$c_i : \phi = \gamma_i + \psi_i(r, \phi), \quad i = 1, 2, \ldots, p, \tag{3}$$

where ψ_i is a 2π-periodic function in ϕ, continuously differentiable and moreover $\psi_i = O(r)$. Using this discussion which makes use of polar transformation, we get

$$\tilde{D}_1 = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_p + \psi_p(r, \phi) < \phi \leq \gamma_1 + 2\pi + \psi_1(r, \phi)\},$$
$$\tilde{D}_i = \{(r, \phi) \mid r \geq 0 \text{ and } \gamma_{i-1} + \psi_{i-1}(r, \phi) < \phi \leq \gamma_i + \psi_i(r, \phi)\}, \quad i = 2, 3, \ldots, p.$$

Let ε be a positive number and $N_\varepsilon(\tilde{D}_i)$ denote the ε-neighborhoods of the regions $\tilde{D}_i$, $i = 1, 2, \ldots, p$.

(P2) Let f_i be a function defined on $N_\varepsilon(\tilde{D}_i)$ and $f_i \in C^{(2)}(N_\varepsilon(\tilde{D}_i))$ for each $i = 1, 2, \ldots, p$.
(P3) $\tau_i \in C^{(2)}(N_\varepsilon(\tilde{D}_i))$, $i = 1, 2, \ldots, p$.
(P4) $f_i(y) = o(\|y\|)$, $i = 1, 2, \ldots, p$.

We shall consider the function $\tilde{f}(y) = A_i y + f_i(y)$ for $y \in \tilde{D}_i$, where the matrix A_i is as described in the assumption (A2). On Ω, we now study the following differential equation associated with (1)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{f}(y), \tag{4}$$

where the function $\tilde{f}(y)$ has discontinuities on the curves c_i, $i = 1, 2, \ldots, p$.

If Ω is sufficiently small, then conditions (A1) and (P1) imply that the curves c_i intersect each other only at the origin, none of them can intersect itself and $\langle \frac{\partial \tilde{\Phi}_i(y)}{\partial y}, \tilde{f}(y)\rangle \neq 0$ for $y \in c_i$, $i = 1, 2, \ldots, p$.

Further, for system (4) if a solution which starts sufficiently close to the origin on a curve c_i with fixed i, then conditions mentioned above imply the continuation of the solution to the curve c_{i+1} or c_{i-1} depending on the direction of the time.

We can utilize polar coordinates and assume that system (4) transforms into an equivalent system of the form

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \tilde{g}(r, \phi), \tag{5}$$

where $\tilde{g}(r, \phi) = \lambda_i r + P_i(r, \phi)$ for $(r, \phi) \in \tilde{D}_i$. The function P_i is 2π-periodic in ϕ, continuously differentiable and $P_i = o(r)$, $i = 1, 2, \ldots, p$.

From the construction, we see that system (5) is a differential equation with discontinuous right-hand side and the discontinuities occur on the curves c_i, $i = 1, 2, \ldots, p$. In almost every area of differential equations, it is common to reduce a given equation into an equivalent form by proper methods. From this point of view, we shall use the B-equivalence method [23,25] which plays the role of a bridge in the passage from differential equations with discontinuous right-hand side to impulsive differential equations.

To reduce the system (5) with discontinuous vector fields into an impulsive differential equation, we redefine the function $\tilde{g}$ in the neighborhoods of the straight lines l_i, which contain the curve c_i. That is to say, we construct a new function g_N which is continuous everywhere except possibly at the points $(r, \phi) \in l_i$. The redefinition will be made at the points which lie between l_i and c_i and belong to the regions D_i or D_{i+1} for each i. Therefore, the construction is performed with minimal possible changes corresponding to the B-equivalence method, which is the main instrument of our investigation.

It is clear from the context that if $i = p$ then $D_{p+1} = D_1$. Using the argument above, we realize the following reconstruction of the domain. We consider the subregions of D_i and D_{i+1}, which are placed between the straight line l_i and the curve c_i. We refer to the subregions $D_i \cap \tilde{D}_{i+1}$ (horizontally shaded regions in Fig. 2) and $D_{i+1} \cap \tilde{D}_i$ (vertically shaded regions in

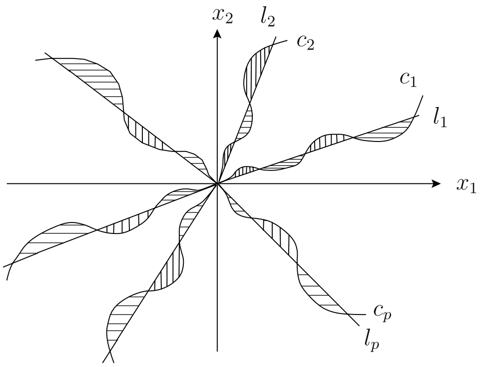

Fig. 2. The domain of the perturbed system (4) near a vertex which unites the curves c_i associated with the straight lines l_i, $i = 1, 2, \dots, p$.

Fig. 2) for all i. We extend the function $\tilde{g}$ from the region $D_i \cap \tilde{D}_{i+1}$ to D_i and from $D_{i+1} \cap \tilde{D}_i$ to D_{i+1} so that the new function g_N and its partial derivatives become continuous up to the angle $\phi = \gamma_i$, $i = 1, 2, \dots, p$. According to all these discussions made for the definition of g_N, we conclude that $g_N(r, \phi) = \lambda_i r + P_i(r, \phi)$ for $(r, \phi) \in D_i$. Now we consider the following differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = g_N(r, \phi). \tag{6}$$

Fix $i \in \{1, 2, \dots, p\}$ and consider a neighborhood of l_i based on the description above. We need to analyze the following three cases:

I. Assume that the point $(r, \gamma_i) \in \tilde{D}_{i+1}$. Let $r^0(\phi) = r(\phi, \gamma_i, \rho)$ be a solution of (5) satisfying $r^0(\gamma_i) = \rho$ and ξ_i be the angle where this solution crosses the curve c_i. We denote a solution of (6) by $r^1(\phi) = r(\phi, \xi_i, r^0(\xi_i))$, $r^1(\xi_i) = r^0(\xi_i)$, on the interval $[\xi_i, \gamma_i]$. By the variation of constant formula, these solutions have the form

$$r^0(\phi) = \exp(\lambda_{i+1}(\phi - \gamma_i))\rho + \int_{\gamma_i}^{\phi} \exp(\lambda_{i+1}(\phi - s))P_{i+1}(r^0(s), s)\mathrm{d}s,$$

$$r^1(\phi) = \exp(\lambda_i(\phi - \xi_i))r^0(\xi_i) + \int_{\xi_i}^{\phi} \exp(\lambda_i(\phi - s))P_i(r^1(s), s)\mathrm{d}s.$$

Now, we define a mapping I_i on the line $\phi = \gamma_i$ into itself as follows

$$\begin{aligned} I_i(\rho) &= r^1(\gamma_i) - \rho = (\exp((\lambda_i - \lambda_{i+1})(\gamma_i - \xi_i)) - 1)\,\rho \\ &\quad + \exp(\lambda_i(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_{i+1}(\xi_i - s))P_{i+1}\mathrm{d}s + \int_{\xi_i}^{\gamma_i} \exp(\lambda_i(\gamma_i - s))P_i\mathrm{d}s. \end{aligned}$$

II. If the point $(r, \gamma_i) \in \tilde{D}_i$, one can find I_i in a similar manner:

$$\begin{aligned} I_i(\rho) &= (\exp((\lambda_i - \lambda_{i+1})(\xi_i - \gamma_i)) - 1)\,\rho \\ &\quad + \exp(\lambda_{i+1}(\gamma_i - \xi_i)) \int_{\gamma_i}^{\xi_i} \exp(\lambda_i(\xi_i - s))P_i\mathrm{d}s + \int_{\xi_i}^{\gamma_i} \exp(\lambda_{i+1}(\gamma_i - s))P_{i+1}\mathrm{d}s. \end{aligned}$$

III. If $(r, \gamma_i) \in c_i$, then $I_i(\rho) = 0$.

Results from [22] imply that the functions I_i, $i = 1, 2, \dots, p$, are continuously differentiable and the Eq. (3) leads us to $I_i = o(\rho)$.

Hereby we construct the following impulsive differential equation

$$\begin{aligned} &\frac{\mathrm{d}\rho}{\mathrm{d}\phi} = g_N(\rho, \phi), \quad \phi \neq \gamma_i, \\ &\Delta\rho\,|_{\phi=\gamma_i} = I_i(\rho). \end{aligned} \tag{7}$$

Let $r(\phi, r_0)$ be a solution of (5), $r(0, r_0) = r_0$, and ξ_i be the meeting angle of this solution with the curve c_i. Denote by $(\xi_i, .\gamma_i]$ the interval $(\xi_i, \gamma_i]$ whenever $\xi_i \leq \gamma_i$ and $[\gamma_i, \xi_i)$ if $\gamma_i < \xi_i$.

Definition 3.1. We shall say that systems (5) and (7) are B-equivalent in Ω if for every solution $r(\phi, r_0)$ of (5) whose trajectory is in Ω for all $\phi \in [0, 2\pi]$ there exists a solution $\rho(\phi, r_0)$ of (7) which satisfies the relation

$$r(\phi, r_0) = \rho(\phi, r_0), \quad \phi \in [0, 2\pi] \setminus \bigcup_{i=1}^{p} (\xi_i, .\gamma_i], \tag{8}$$

and, conversely, for every solution $\rho(\phi, r_0)$ of (7) whose trajectory is in Ω, there exists a solution $r(\phi, r_0)$ of (5) which satisfies (8).

From the discussion above and the construction of the impulsive system (7) with impulse actions at fixed angles, it follows that for sufficiently small Ω, solution $r(\phi, r_0)$ of (5) whose trajectory is in Ω for all $\phi \in [0, 2\pi]$ takes the same values with the exception of the oriented intervals $(\xi_i, .\gamma_i]$ as the solution $\rho(\phi, r_0)$, $\rho(0, r_0) = r_0$, of (7). Hence, systems (5) and (7) are B-equivalent in the sense of the Definition 3.1. Moreover, solutions of (5) exist in the neighborhood Ω, they are continuous and have discontinuities in the derivative on the curves c_i. Correspondingly, a solution of system (4) for any initial value is continuous, continuously differentiable except possibly at the moments when the trajectories intersect the curves c_i, and it is unique.

Theorem 3.1. *Suppose* (A1)–(A2), (P1)–(P4) *are satisfied and* $q < 1$ $(q > 1)$. *Then the origin is a stable (unstable) focus of* (4).

Proof. Let $r(\phi, r_0)$ be the solution of (5) with $r(0, r_0) = 0$ and $\rho(\phi, r_0)$, $\rho(0, r_0) = r_0$, be the solution of (7). For the sake of simplicity, we shall use the notations $P_i = P_i(\rho(s, r_0), s)$ and $I_i = I_i(\rho(\gamma_i, r_0))$, $i = 1, 2, \ldots, p$.

On the interval $\phi \in [0, \gamma_1]$, we have

$$\rho(\phi, r_0) = \exp(\lambda_1 \phi) r_0 + \int_0^{\phi} \exp(\lambda_1(\phi - s)) P_1 \mathrm{d}s.$$

For any i, $2 \le i \le p$, the solution $\rho(\phi, r_0)$ of (7) on $(\gamma_{i-1}, \gamma_i]$ is given by

$$\begin{aligned}
\rho(\phi, r_0) = {} & \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \lambda_{i-1}\theta_{i-1} + \cdots + \lambda_2\theta_2 + \lambda_1\gamma_1\right) r_0 \\
& + \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \cdots + \lambda_2\theta_2 + \lambda_1\gamma_1\right) \int_0^{\gamma_1} \exp(-\lambda_1 s) P_1 \mathrm{d}s \\
& + \sum_{k=2}^{i-1} \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \cdots + \lambda_{k+1}\theta_{k+1} + \lambda_k\gamma_k\right) \int_{\gamma_{k-1}}^{\gamma_k} \exp(-\lambda_k s) P_k \mathrm{d}s \\
& + \int_{\gamma_{i-1}}^{\phi} \exp\left(\lambda_i(\phi - s)\right) P_i \mathrm{d}s + \sum_{k=2}^{i} \exp\left(\lambda_i(\phi - \gamma_{i-1}) + \lambda_{i-1}\theta_{i-1} + \cdots + \lambda_k\theta_k\right) I_{k-1}.
\end{aligned}$$

For $\phi \in (\gamma_p, 2\pi]$, system (7) admits the solution

$$\rho(\phi, r_0) = \exp\left(\lambda_1(\phi - \gamma_p)\right)\left(\rho(\gamma_p, r_0) + I_p\right) + \int_{\gamma_p}^{\phi} \exp\left(\lambda_1(\phi - s)\right) P_1 \mathrm{d}s.$$

Using the differentiable dependence of solutions of impulse systems on parameters [24] and the results from [22], we can conclude that the solution $\rho(\phi, r_0)$ is differentiable in r_0 and $\frac{\partial \rho(\phi, r_0)}{\partial r_0}\big|_{(\phi, r_0) = (2\pi, 0)} = q$. Since systems (5) and (7), correspondingly (4) and (7), are B-equivalent, we derive

$$\frac{\partial r(\phi, r_0)}{\partial r_0}\Big|_{(\phi, r_0) = (2\pi, 0)} = q,$$

which completes the proof. □

4. The focus–center problem

If $q = 1$, then we have the critical case and the origin is either a focus or a center for system (4). In what follows, we solve this problem of distinguishing between the focus and the center.

We assume that f_i and τ_i, $i = 1, 2, \ldots, p$, are analytic functions in $N_\varepsilon(\tilde{D}_i)$. Then for sufficiently small ρ, the solution $\rho(\phi, r_0)$ of (7) satisfying $\rho(0, r_0) = r_0$ has the expansion [23]

$$\rho(\phi, r_0) = \sum_{j=0}^{\infty} \rho_j(\phi) r_0^j, \tag{9}$$

for all $\phi \in [0, 2\pi]$. From the expansion (9), it can be easily seen that $\rho_1(0) = 1$, $\rho_i(0) = 0$ for all $i = 0, 2, 3, 4, \ldots$, and $\rho_0(\phi) = 0$. The coefficient $\rho_1(\phi)$ with $\rho_1(0) = 1$ is the solution of the system

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}\phi} = g(\rho_1), \tag{10}$$

where g is the function defined in system (2). It is clear that $\rho_1(2\pi) = q = 1$. We use the notation $k_j = \rho_j(2\pi), j = 2, 3, \ldots$. For the solution $\rho(\phi, r_0)$ of (7), we construct the Poincaré return map

$$\rho(2\pi, r_0) = qr_0 + \sum_{j=2}^{\infty} k_j r_0^j.$$

In the critical case, the sign of the first nonzero element of the sequence k_j determines what type of a singular point the origin is. Moreover, for all $i = 1, 2, \ldots, p$, we have

$$P_i(\rho, \phi) = \sum_{j=2}^{\infty} P_{ij}(\phi)\rho^j, \tag{11}$$

and

$$I_i(\rho) = \sum_{j=2}^{\infty} I_{ij}\rho^j. \tag{12}$$

The existence of the expansions (11) and (12) has been proved in [23]. By means of (11) and (12), one can derive that the coefficients $\rho_j(\phi)$ with $\rho_j(0) = 0, j = 2, 3, \ldots$, are solutions of the following impulsive system

$$\begin{aligned} \frac{d\rho_j}{d\phi} &= h(\rho_j, \phi), \quad \phi \neq \gamma_i, \\ \Delta\rho_j|_{\phi=\gamma_i} &= W_{ij}, \end{aligned} \tag{13}$$

where $h(\rho_j, \phi) = \lambda_i\rho_j + Q_{ij}(\phi)$ if $(\rho_j, \phi) \in D_i, i = 1, 2, \ldots, p$. From the differential part of (7) and the expansion (11), one can evaluate for any i, $1 \leq i \leq p$,

$$Q_{i2}(\phi) = P_{i2}(\phi)\rho_1^2(\phi), \qquad Q_{i3}(\phi) = 2P_{i2}(\phi)\rho_1(\phi)\rho_2(\phi) + P_{i3}(\phi)\rho_1^3(\phi)$$

and $Q_{ij}(\phi)$ for $j = 4, 5, \ldots$, can be determined similarly. Further, the constants W_{ij} in (13) can be found from the impulsive part of (7) and the expansion (12). For instance,

$$W_{i2} = I_{i2}\rho_1^2(\gamma_i), \qquad W_{i3} = 2I_{i2}\rho_1(\gamma_i)\rho_2(\gamma_i) + I_{i3}\rho_1^3(\gamma_i),$$

and W_{ij} can be evaluated, for $j = 4, 5, \ldots$, in the same manner.

As $k_j = \rho_j(2\pi)$, by solving the system (13) one can evaluate $k_j, j = 2, 3, \ldots$, which are the coefficients in the expansion of the Poincaré return map $\rho(2\pi, r_0)$:

$$\begin{aligned} k_j = & \int_0^{\gamma_1} \exp(-\lambda_1 s)Q_{1j}ds + \int_{\gamma_p}^{2\pi} \exp\left(\lambda_1(2\pi - s)\right) Q_{1j}ds \\ & + \sum_{i=2}^{p} \exp\left(\lambda_1(2\pi - \gamma_p) + \cdots + \lambda_{i+1}\theta_{i+1} + \lambda_i\gamma_i\right) \int_{\gamma_{i-1}}^{\gamma_i} \exp(-\lambda_i s)Q_{ij}ds \\ & + \sum_{i=2}^{p} \exp\left(\lambda_1(2\pi - \gamma_p) + \lambda_p\theta_p + \cdots + \lambda_i\theta_i\right) W_{i-1,j} + \exp\left(\lambda_1(2\pi - \gamma_p)\right) W_{pj}. \end{aligned} \tag{14}$$

From the expansion of $\rho(2\pi, r_0)$ and (14), it immediately follows that the following assertion is valid.

Lemma 4.1. *Let $q = 1$ and the first nonzero element of the sequence $k_j, j = 2, 3, \ldots$, be negative (positive). Then the origin is a stable (unstable) focus of (7). If $k_j = 0$ for all $j \geq 2$, then the origin is a center for system (7).*

Since systems (5) and (7), correspondingly (4) and (7), are B-equivalent, we have proved the following theorem.

Theorem 4.1. *Let $q = 1$ and the first nonzero element of the sequence $k_j, j = 2, 3, \ldots$, be negative (positive). Then the origin is a stable (unstable) focus of (4). If $k_j = 0$ for all $j \geq 2$, then the origin is a center for system (4).*

5. Hopf bifurcation

In this section, we first introduce the system

$$\frac{dz}{dt} = \hat{f}(z, \mu) \tag{15}$$

where $\hat{f}(z, \mu) = A_i z + f_i(z) + \mu F_i(z, \mu)$ for $z \in \tilde{D}_i(\mu) \subset \mathbb{R}^2$ for analysis, and then we will describe it in detail with the help of the following assumptions.

(H1) Let $\{c_i(\mu)\}_{i=1}^{p}$ be a collection of curves in Ω which start at the origin and are given by the equations $\langle a^i, z\rangle + \tau_i(z) + \mu\kappa_i(z,\mu) = 0$, $i = 1, 2, \ldots, p$.
(H2) Let $\{l_i(\mu)\}_{i=1}^{p}$ be a union of half-lines which start at the origin and are defined by $\langle a^i + \mu\frac{\partial\kappa_i(0,\mu)}{\partial z}, z\rangle = 0$, $i = 1, 2, \ldots, p$. Denote by $\gamma_i(\mu)$ the angles of the lines $l_i(\mu)$, $i = 1, 2, \ldots, p$.

Similar to the construction of the regions D_i and $\tilde{D_i}$, we set for $\mu \in (-\mu_0, \mu_0)$ and $i = 2, 3, \ldots, p$:

$$\tilde{D_1}(\mu) = \{(r,\phi,\mu) \mid r \geq 0,\ \gamma_p(\mu) + \Psi_p < \phi \leq \gamma_1(\mu) + 2\pi + \Psi_1\},$$
$$\tilde{D_i}(\mu) = \{(r,\phi,\mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) + \Psi_{i-1} < \phi \leq \gamma_i(\mu) + \Psi_i\},$$
$$D_1(\mu) = \{(r,\phi,\mu) \mid r \geq 0,\ \gamma_p(\mu) < \phi \leq \gamma_1(\mu) + 2\pi\},$$
$$D_i(\mu) = \{(r,\phi,\mu) \mid r \geq 0,\ \gamma_{i-1}(\mu) < \phi \leq \gamma_i(\mu)\},$$

where functions $\Psi_i = \Psi_i(r,\phi,\mu)$ are 2π-periodic in ϕ, continuously differentiable, $\Psi_i = O(r)$, $i = 1, 2, \ldots, p$, and they can be defined applying a similar technique used in the construction of Eq. (3).

(H3) $F_i\colon N_\varepsilon(\tilde{D_i}(\mu)) \times (-\mu_0, \mu_0) \to \mathbb{R}^2$ and κ_i are analytical functions both in z and μ in the ε-neighborhood of their domains.
(H4) $F_i(0,\mu) = 0$ and $\kappa_i(0,\mu) = 0$ hold uniformly for each i and $\mu \in (-\mu_0, \mu_0)$.
(H5) The matrices A_i, the functions f_i, τ_i and the constant vectors a^i correspond to the ones described in systems (1) and (4).

Besides the system (15), we need the equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \hat{f}_z(0,\mu)z, \tag{16}$$

where $\hat{f}_z(0,\mu) = A_i + \mu\frac{\partial F_i(0,\mu)}{\partial z}$ whenever $z \in D_i(\mu)$.
In polar coordinates, system (15) reduces to

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \hat{g}(r,\phi,\mu), \tag{17}$$

where $\hat{g}(r,\phi,\mu) = \lambda_i(\mu)r + P_i(r,\phi,\mu)$ if $(r,\phi,\mu) \in \tilde{D_i}(\mu)$.
Let the following impulse system

$$\begin{aligned}&\frac{\mathrm{d}\rho}{\mathrm{d}\phi} = \hat{g}_N(\rho,\phi,\mu), \quad \phi \neq \gamma_i(\mu),\\ &\Delta\rho\,|_{\phi=\gamma_i(\mu)} = I_i(\rho,\mu)\end{aligned} \tag{18}$$

be B-equivalent to (17), where $\hat{g}_N$ stands for the extension of $\hat{g}$ as we described in Section 2. That is, $\hat{g}_N(\rho,\phi,\mu) = \lambda_i(\mu)\rho + P_i(\rho,\phi,\mu)$ for $(\rho,\phi,\mu) \in D_i(\mu)$. We know that the function $\hat{g}_N$ and its partial derivatives become continuous up to the angle $\phi = \gamma_i(\mu)$ for $i = 1, 2, \ldots, p$. The function $I_i(\rho,\mu)$, for each $i = 1, 2, \ldots, p$ can be defined in the same way as done for $I_i(\rho)$.

Using a similar argument as in (1), we can obtain for system (16) that

$$q(\mu) = \exp\left(\sum_{i=1}^{p}\lambda_i(\mu)\theta_i(\mu)\right).$$

The last expression plays an important rule to establish the theorem on the bifurcation of periodic solutions as stated below. To prove this assertion we apply the technique of paper [22].

Theorem 5.1. *Let $q(0) = 1$, $q'(0) \neq 0$ and the origin be a focus for (4). Then, for sufficiently small r_0, there exists a unique continuous function $\mu = \delta(r_0)$, $\delta(0) = 0$, such that the solution $r(\phi, r_0, \delta(r_0))$ of (17) is periodic with period 2π. Moreover, the closed trajectory is stable (unstable) if the origin of (4) is a stable (unstable) focus. The period of the corresponding periodic solution of (15) is $T = \sum_{i=1}^{p}\frac{\theta_i}{\beta_i} + o(|\mu|)$.*

Proof. Let $\rho(\phi, r_0, \mu)$ be the solution of (18) such that $\rho(0, r_0, \mu) = r_0$. To exclude the trivial solution, we consider $r_0 > 0$. The theorem of analyticity of solutions [23] imply that

$$\rho(2\pi, r_0, \mu) = \sum_{j=1}^{\infty}k_j(\mu)r_0^j,$$

where $k_j(\mu) = \sum_{i=0}^{\infty}k_{ji}\mu^i$. Since $k_1(\mu) = q(\mu)$, we have by the hypotheses of the theorem that $k_{10} = q(0) = 1$ and $k_{11} = q'(0) \neq 0$. For the existence of a periodic solution we require that $\rho(2\pi, r_0, \mu) = r_0$. Now we define $\mathcal{F}(r_0,\mu) = \rho(2\pi, r_0, \mu) - r_0$. Then, it can be derived that

$$\mathcal{F}(r_0,\mu) = q'(0)\mu r_0 + \sum_{j=2}^{\infty}k_{j0}r_0^j + \sum_{i+j\geq 3}k_{ji}\mu^i r_0^j,$$

where $i, j \in \mathbb{N}$ in the second summation. We call $\mathcal{F}(r_0, \mu) = 0$ as the bifurcation equation. If we cancel by r_0, we obtain the equation

$$\mathcal{H}(r_0, \mu) = 0, \tag{19}$$

where

$$\mathcal{H}(r_0, \mu) = q'(0)\mu + \sum_{j=2}^{\infty} k_{j0} r_0^{j-1} + \sum_{i+j \geq 2} k_{j+1,i} \mu^i r_0^j.$$

In the second summation of the last equation, we have $i = 1, 2 \ldots,$ and $j = 0, 1, \ldots$. Since $\mathcal{H}(0,0) = 0$ and $\frac{\partial \mathcal{H}(0,0)}{\partial \mu} = q'(0) \neq 0$, one can say by the implicit function theorem that for sufficiently small r_0 there exists a function $\mu = \delta(r_0)$ such that $\rho(\phi, r_0, \delta(r_0))$ is a periodic solution.

We assume without loss of generality that $k_{j0} = 0$ for $j = 2, 3, \ldots, l-1$ and $k_{l0} \neq 0$. Then we can obtain from (19) that

$$\delta(r_0) = -\frac{k_{l0}}{q'(0)} r_0^{l-1} + \sum_{i=l}^{\infty} \delta_i r_0^i. \tag{20}$$

If we analyze the Eq. (20), we can conclude that the bifurcation of periodic solutions exists if a stable (unstable) focus for $\mu = 0$ becomes unstable (stable) for $\mu \neq 0$.

Let $\rho(\phi) = \rho(\phi, \bar{r_0}, \bar{\mu})$ be a periodic solution of (18). This periodic solution is a limit cycle if $\frac{\partial \mathcal{F}(\bar{r_0}, \bar{\mu})}{\partial r_0} < 0$. Assuming that the first nonzero element k_{l0} of the sequence $k_{j0}, j \geq 2$, is negative and using (20), we get

$$\frac{\partial \mathcal{F}(\bar{r_0}, \bar{\mu})}{\partial r_0} = (l-1) k_{l0} \bar{r_0}^{\,l-1} + \mathcal{G}(r_0),$$

where $\mathcal{G}$ starts with a member whose order is not less than l. Thus, it implies that $\frac{\partial \mathcal{F}(\bar{r_0}, \bar{\mu})}{\partial r_0} < 0$.

Since (17) and (18) are B-equivalent systems, the proof is completed. $\square$

6. An example

To be convenient, in the following example we use the corresponding notations that are adopted above.

Example 6.1. Let $c_1(\mu)$ and $c_2(\mu)$ be the curves defined by $z_2 = \frac{1}{\sqrt{3}} z_1 + (1+\mu) z_1^3, z_1 > 0$ and $z_2 = \sqrt{3} z_1 + z_1^5 + \mu z_1^2$, $z_1 < 0$, respectively. We take

$$A_1 = \begin{bmatrix} -0.7 & -2 \\ 2 & -0.7 \end{bmatrix}, \qquad f_1(z) = \begin{bmatrix} z_1 \sqrt{z_1^2 + z_2^2} \\ z_2 \sqrt{z_1^2 + z_2^2} \end{bmatrix}, \qquad F_1(z, \mu) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} 0.5 & -2 \\ 2 & 0.5 \end{bmatrix}, \qquad f_2(z) = \begin{bmatrix} -2z_1 \sqrt{z_1^2 + z_2^2} \\ -2z_2 \sqrt{z_1^2 + z_2^2} \end{bmatrix}, \qquad F_2(z, \mu) = \begin{bmatrix} -z_1 \\ -z_2 \end{bmatrix}.$$

After these preparations, we consider the system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \hat{f}(z, \mu) \tag{21}$$

where $\hat{f}(z, \mu) = A_i z + f_i(z) + \mu F_i(z, \mu)$ if $z \in \tilde{D_i}(\mu), i = 1, 2$. Here $\tilde{D_1}(\mu)$ denotes the region situated between the curves $c_1(\mu)$ and $c_2(\mu)$, which contains the fourth quadrant. $\tilde{D_1}(\mu)$ is the region between $c_1(\mu)$ and $c_2(\mu)$ containing the second quadrant.

Since $q = 1$, by Theorem 2.1 the origin is a center for the nonperturbed system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

where $f(x) = A_i x$ whenever $x \in D_i$, $i = 1, 2$ as shown in Fig. 3. Here D_1 and D_2 are the regions between the half straight lines $l_1 : z_2 = \frac{1}{\sqrt{3}} z_1, z_1 > 0$ and $l_2 : z_2 = \sqrt{3} z_1, z_1 < 0$, which contain the fourth and second quadrants, respectively.

One can see that $l_1(\mu)$ ($l_2(\mu)$) coincides with l_1 (l_2). Hence, $\gamma_1 = \gamma_1(\mu) = \frac{\pi}{6}$ and $\gamma_2 = \gamma_2(\mu) = \frac{4\pi}{3}$. Using the given information, we obtain

$$q(\mu) = \exp\left(-\frac{\pi}{6}\mu\right), \qquad q(0) = 1, \qquad q'(0) = -\frac{\pi}{6} \neq 0.$$

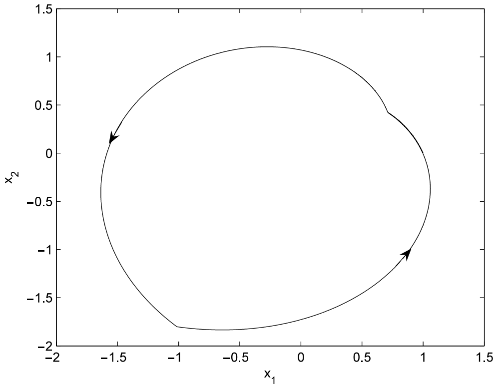

Fig. 3. The simulation result showing the existence of a center for the nonperturbed system.

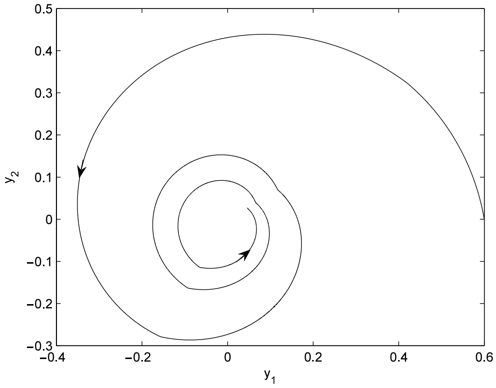

Fig. 4. The simulation result showing the existence of a stable focus for the perturbed system ($\mu = 0$).

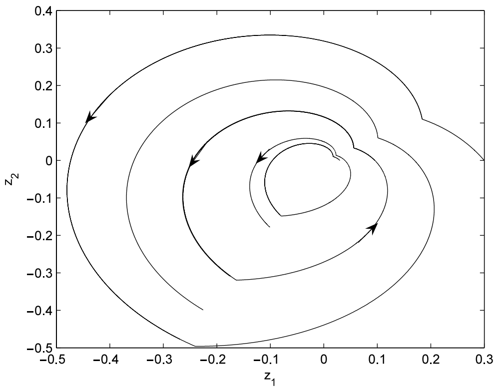

Fig. 5. The simulation result showing the existence of a limit cycle for system (21).

Moreover, for the associated system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{f}(y),$$

where $\tilde{f}(y) = A_i y + f_i(y)$ whenever $y \in \tilde{D}_i$, $i = 1, 2$, it follows from Theorem 4.1 that the origin is a stable focus as $k_2 < 0$ for the perturbed system (see Fig. 4). Here $\tilde{D_1}$ and $\tilde{D_2}$ are the regions between the curves $c_1 : z_2 = \frac{1}{\sqrt{3}} z_1 + z_1^3, z_1 > 0$ and $c_2 : z_2 = \sqrt{3} z_1 + z_1^5, z_1 < 0$, which contain the fourth and second quadrants, respectively.

From Fig. 5, which is simulated for $\mu = -0.8$, we see that the trajectories approach to the periodic solution from interior and exterior. That is to say, system (21) has a limit cycle with period $\approx \pi$.

7. Conclusion

Hopf bifurcation for smooth systems is characterized by a pair of complex conjugate eigenvalues of the linearized system. It is well known that it is not the case for systems of differential equations with discontinuities. Although the system specified in (15) together with the assumption (A2) reflects a special class of such systems, it is worthwhile to develop a technique for the investigation of bifurcation problems as they exhibit complicated bifurcation phenomena. Further, the problem can be generalized by taking the matrices A_i, $i = 1, 2, \ldots, p$, not only of focus type in all subregions but also of other types, e.g., they may be hyperbolic with real eigenvalues. Clearly, this problem can be analyzed in a similar way when it is required by concrete applications in mechanics, electronics, biology etc.

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References

[1] A.A. Andronow, C.E. Chaikin, Theory of Oscillations, Princeton Univ. Press, Princeton, 1949.
[2] M. Kunze, T. Küpper, Qualitative bifurcation analysis of a non-smooth friction-oscillator model, Z. Angew. Math. Phys. 48 (1997) 87–101.
[3] M. Kunze, Non-smooth dynamical systems, in: Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, New York, 2000.
[4] N. Minorsky, Nonlinear Oscillations, Van Nostrand, New York, 1962.
[5] A.F. Filippov, Differential Equations with Discontinuous Righthand Sides, Kluwer Academic Publishers, Dordrecht, 1988.
[6] C. Henry, Differential equations with discontinuous righthand side for planning procedure, J. Econom. Theory 4 (1972) 541–551.
[7] N.N. Bautin, E.A. Leontovich, Methods and Rules for the Qualitative Study of Dynamical Systems on the Plane, Nauka, Moscow, 1990 (in Russian).
[8] T.R. Blows, N.G. Lloyd, The number of limit cycles of certain polynomial differential equations, Proc. Roy. Soc. Edinburgh Sect. 98A (1984) 215–239.
[9] S.-N. Chow, J.K. Hale, Methods of Bifurcation Theory, Springer, New York, Heidelberg, Berlin, 1982.
[10] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, New York, 1983.
[11] J.E. Marsden, M. McCracken, The Hopf bifurcation and its applications, in: Applied Mathematical Sciences, Springer, New York, Berlin, 1976.
[12] B. Coll, A. Gasull, R. Prohens, Degenerate Hopf bifurcations in discontinuous planar systems, J. Math. Anal. Appl. 253 (2001) 671–690.
[13] T. Küpper, S. Moritz, Generalized Hopf bifurcation for non-smooth planar systems, Phil. Trans. R. Soc. Lond. 359 (2001) 2483–2496.
[14] J. Llibre, E. Ponce, Bifurcation of a periodic orbit from infinity in planar piecewise linear vector fields, Nonlinear Anal. 36 (1999) 623–653.
[15] H.E. Nusse, J.A. Yorke, Border-collision bifurcations including period two to period three for piecewise smooth systems, Physica D 57 (1992) 39–57.
[16] H.E. Nusse, J.A. Yorke, Border-collision bifurcations for piecewise smooth one-dimensional maps, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 5 (1995) 189–207.
[17] Y. Zou, T. Küpper, Generalized Hopf bifurcation emanated from a corner for piecewise smooth planar systems, Nonlinear Anal. 62 (2005) 1–17.
[18] Y. Zou, T. Küpper, W.-J. Beyn, Generalized Hopf bifurcation for planar Filippov systems continuous at the origin, J. Nonlinear Sci. 16 (2006) 159–177.
[19] H. Dankowicz, A.B. Nordmark, On the origin and bifurcations of stick-slip oscillations, Physica D 136 (2000) 280–302.
[20] M.I. Feigin, Doubling of the oscillation period with C-bifurcations in piecewise-continuous systems, J. Appl. Math. Mech. 34 (1970) 822–830.
[21] M.I. Feigin, On the structure of C-bifurcation boundaries of piecewise-continuous systems, J. Appl. Math. Mech. 42 (1978) 820–829.
[22] M.U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, Nonlinear Anal. 60 (2005) 163–178.
[23] M. Akhmetov, Asymptotic representation of solutions of regularly perturbed systems of differential equations with a nonclassical right-hand side, Ukrainian Math. J. 43 (1991) 1209–1214.
[24] M. Akhmetov, N.A. Perestyuk, Differential properties of solutions and integral surfaces of nonlinear impulse systems, Differential Equations 28 (1992) 445–453.
[25] M. Akhmetov, On the expansion of solutions to differential equations with discontinuous right-hand side in a series in initial data and parameters, Ukrainian Math. J. 45 (1993) 786–789.
[26] E. Akalın, M.U. Akhmet, The principles of B-smooth discontinuous flows, Math. Comput. Simul. 49 (2005) 981–995.