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Exponentially dichotomous linear systems of differential equations with piecewise constant argument

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Abstract

We consider differential equations with piecewise constant argument of generalized type. It is the first time, an attention is given to the exponential dichotomy of linear systems. Bounded, almost periodic and periodic solutions and their stability are discussed. The study is made in such a way that further construction of the theory will follow for ordinary differential equations. The results are illustrated by examples.

1 Introduction and Preliminaries

Differential equations with piecewise constant argument (EPCA) were proposed for investigations in [1, 2] by founders of the theory. In papers [3]- [22] and many others, the traditional method of investigation has been effectively utilized for various interesting problems of theory and applications. The traditional method means that the constant argument is assumed to be a multiple of the *greatest integer function*, and analysis is based on reduction to *discrete equations*. In fact, the simple type of constancy and the reduction are strongly related to each other, since it is respectively easy to reduce to discrete equation systems with this type of argument.

A new class of differential equations (EPCAG) was introduced in [23], and then developed in [24]- [37]. Extended information about these systems can be found in book [24]. They contain EPCA as a subclass. In paper [23], we not only *generalized* the piecewise constant argument, but proposed to investigate the newly introduced systems by reducing them to *integral equations*. This innovation became very effective, and

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there are two main reasons for that. Firstly, it is feasible, now, to investigate systems, which are essentially non-linear. More precisely, non-linear with respect to values of solutions at discrete moments of time. While in EPCA values of solutions at the discrete moments appear only linearly [18]. Secondly, we can analyze existence, uniqueness and stability of solutions with arbitrary initial moments, not only the discrete moments of time. Thus, we have deepen the analysis insight significantly. Further our proposals were developed in papers [25]- [37] and [38, 39] from theoretical point of view, and for applications [31, 32, 40]. In the present paper we give more lights on the subject. We consider linear homogeneous as well as non-homogeneous systems. Basic attributes of linear systems, dimension of the space of solutions, fundamental matrix of solutions, state transition matrix, general solution, periodic and almost periodic solutions are carefully discussed. Exponential dichotomy for EPCAG is introduced with examples. Existence of bounded solutions, almost periodic and periodic solutions are under discussion. It is clear that many questions, which relate to linear and quasilinear systems, will be solved later by using results of this paper.

Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $||\cdot||$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Fix two real-valued sequences $\theta_i, \zeta_i, i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}, \theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$.

We shall consider the following two equations, [24, 27],

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)), \tag{1}$$

and

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)) + f(t), \tag{2}$$

where $z \in \mathbb{R}^n, t \in \mathbb{R}, \gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1}), i \in \mathbb{Z}$.

We assume that the coefficients $A_0(t), A_1(t)$ are continuous on $\mathbb{R}$, $n \times n$, real valued matrices, the function $f(t)$ is continuous. In our paper we assume that the solutions of the equation are *continuous functions*. But the deviating function $\gamma(t)$ is discontinuous. Hence, in general, the right-hand sides of (1) and (2) have discontinuities at moments $\theta_i, i \in \mathbb{Z}$. Summarizing, we consider the solutions of the equations as functions, which are continuous and continuously differentiable within intervals $[\theta_i, \theta_{i+1}), i \in \mathbb{Z}$.

We use the following definition, which is a version of a definition from [18], modified for our general case.

Definition 1. [24, 27, 28] A continuous function $z(t)$ is a solution of (1)((2)) on $\mathbb{R}$ if:

(i) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $\theta_i, i \in \mathbb{Z}$, where the one-sided derivatives exist;

(ii) the equation is satisfied for $z(t)$ on each interval $(\theta_i, \theta_{i+1}), i \in \mathbb{Z}$, and it holds for the right derivative of $z(t)$ at the points $\theta_i, i \in \mathbb{Z}$.

Let I be the $n \times n$ identity matrix. Denote by $X(t,s), X(s,s) = I, s \in \mathbb{R}$, the fundamental matrix of solutions of the system

$$x'(t) = A_0(t)x(t), \tag{3}$$

which is associated with systems (1) and (2). We introduce a matrix-function $M_i(t), i \in \mathbb{Z}$, [24, 27],

$$M_i(t) = X(t, \zeta_i) + \int_{\zeta_i}^{t} X(t,s)A_1(s)ds,$$

useful in what follows. From now on we make the assumption:

- For every fixed $i \in \mathbb{Z}$, $\det[M_i(t)] \neq 0$, $\forall t \in [\theta_i, \theta_{i+1}]$.

We shall call this property, the *regularity condition*.

Theorem 1. *[27] For every $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution $z(t) = z(t, t_0, z_0)$ of (1) in the sense of Definition 1 such that $z(t_0) = z_0$ if and only if the regularity condition is valid.*

The last theorem arranges the correspondence between points $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ and the solutions of (1) in the sense of Definition 1, and there exists no solution of the equation out of the correspondence. Using this assertion we can say that the definition of the IVP for EPCAG is similar to the problem for ordinary differential equations. Particularly, the dimension of the space of all solutions is n. Hence, the investigation of problems considered in our paper does not need to be supported by results of theory of functional differential equations [11], [41], despite the fact EPCAG are equations with deviated arguments.

System (1) is a differential equation with a delay argument. That is why it is reasonable to suppose that the initial "interval" must consist of more than one point. The following arguments show that in our case we need only one initial moment. Indeed, assume that (t_0, z_0) is fixed, and $\theta_i \leq t_0 < \theta_{i+1}$ for a fixed $i \in \mathbb{Z}$. We suppose that $t_0 \neq \zeta_i$. The solution satisfies, on the interval $[\theta_i, \theta_{i+1}]$, the following functional differential equation

$$z'(t) = A_0(t)z + A_1(t)z(\zeta_i). \tag{4}$$

Formally we need the pair of initial points (t_0, z_0) and $(\zeta_i, z(\zeta_i))$ to proceed with the solution, but since $z_0 = M_i(t_0)z(\zeta_i)$, where matrix $M_i(t_0)$ is nonsingular, we can say that the initial condition $z(t_0) = z_0$ is sufficient to define the solution.

Theorem 1 implies that the set of the solutions of (1) is an $n-$dimensional linear space. Hence, for a fixed $t_0 \in \mathbb{R}$ there exists a fundamental matrix of solutions of (1), $Z(t) = Z(t, t_0), Z(t_0, t_0) = I$, such that

$$\frac{dZ}{dt} = A_0(t)Z(t) + A_1(t)Z(\gamma(t)).$$

Without loss of generality, assume that $\theta_i < t_0 < \zeta_i$ for a fixed $i \in \mathbb{Z}$, and define the matrix for increasing t, [24, 27],

$$Z(t) = M_l(t)\Big[\prod_{k=l}^{i+1} M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\Big]M_i^{-1}(t_0), \tag{5}$$

if $t \in [\theta_l, \theta_{l+1}]$, for arbitrary $l > i$.

Similarly, if $\theta_j \leq t \leq \theta_{j+1} \leq \ldots \leq \theta_i \leq t_0 \leq \theta_{i+1}$, then

$$Z(t) = M_j(t)\Big[\prod_{k=j}^{i-1} M_k^{-1}(\theta_{k+1})M_{k+1}(\theta_{k+1})\Big]M_i^{-1}(t_0). \tag{6}$$

One can easily see that

$$Z(t, s) = Z(t)Z^{-1}(s),\ t,\ s \in \mathbb{R}, \tag{7}$$

and a solution $z(t), z(t_0) = z_0, (t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$, of (1) is equal to

$$z(t) = Z(t, t_0)z_0,\ t \in \mathbb{R}. \tag{8}$$

The last formulas, (5)-(8), have been obtained in [27], and are of exceptional importance for our theory of linear systems. It is well known that linear systems with constant coefficients are the source for all concepts of linear systems theory, and they are not EPCAG, since EPCAG are not even smooth. So, we propose to consider system (1) with periodic coefficients as a simplest linear system with piecewise constant argument. Let us describe details in the next example.

Example 1. Assume that there are two numbers, $\omega \in \mathbb{R}, p \in \mathbb{Z}$, such that $\theta_{k+p} = \theta_k + \omega, \zeta_{k+p} = \zeta_k + \omega, k \in \mathbb{Z}$. Then denote by Q the product $\prod_{k=1}^{p} G_k$, where matrices G_k are equal to $M_k^{-1}(\theta_k)M_{k-1}(\theta_k), k \in \mathbb{Z}$. We call the matrix Q, the monodromy matrix, and eigenvalues of the matrix, $\rho_j, j = 1,2,\ldots,n$, multipliers. Let us have some benefits from these definitions. From conditions stated above it follows that, if all multipliers are less than one in absolute value, then there exist numbers, $R \geq 1, \alpha > 0$, such that $\|Z(t,s)\| \leq R\mathrm{e}^{-\alpha(t-s)}, t \geq s$. More exactly, if $\beta = \frac{1}{\omega}\max_i \ln|\rho_i|$, then for arbitrary $\varepsilon > 0$, there exists a number $R(\varepsilon) \geq 1$, such that $\|Z(t,s)\| \leq R(\varepsilon)\mathrm{e}^{-(\beta+\varepsilon)(t-s)}, t \geq s$. It is obvious, that with $|\rho_i| \neq 1$, for all i, we have so called a hyperbolic homogeneous equation.

The last example shows that for (1), one can introduce the concept of exponential dichotomous system.

2 Exponential dichotomy

We say that system (1) satisfies exponential dichotomy, if there exists a projection P and positive constants $\sigma_1, \sigma_2, K_1, K_2$, such that

$$||Z(t)PZ^{-1}(s)|| \leq K_1 \exp(-\sigma_1(t-s)), t \geq s,$$
$$||Z(t)(I-P)Z^{-1}(s)|| \leq K_2 \exp(\sigma_2(t-s)), t \leq s.$$

It is not an easy task to provide examples of exponentially dichotomous systems, since there are no linear EPCAG with constant coefficients, generally. Nevertheless, we can provide in our paper advanced examples.

So, let us pay attention to the periodic system that has been discussed above. Assume that the monodromy matrix, Q, admits no multipliers on the unit circle. More exactly, suppose that k multipliers are inside and $n-k$ of them are outside of the unit disc. It is easy to see that there exists a $k-$dimensional subspace of solutions tending to zero uniformly and exponentially as $t \to \infty$. As well as there exits a $n-k-$dimensional subspace of solutions tending to infinity uniformly and exponentially as $t \to \infty$. On the basis of these observations we can repeat discussions of [42], pages $10-12$, to prove that our periodic system is exponentially dichotomous.

Example 2. Consider sequences of scalars $b_i, \theta_i, i \in \mathbb{Z}$, which satisfy $b_{i+p} = b_i, \theta_{i+p} = \theta_i + \omega, i \in \mathbb{Z}$, for some positive $\omega \in \mathbb{Z}, p \in \mathbb{Z}$. Define the following EPCAG,

$$x' = bx(\gamma(t)), \tag{9}$$

where $\gamma(t) = \theta_i$ if $\theta \leq t < \theta_{i+1}$. One can find that $Q = \prod_{i=p}^{1}[1 + b(\theta_i - \theta_{i-1})]$. Let us, give some analysis by using the last expression. It is seen that the zero solution of the equation is uniformly exponentially stable if, for example, $-1 < 1 + b(\theta_i - \theta_{i-1}) < 1, i = 1,2,\ldots,p$. That is, if $\frac{-2}{\theta_i - \theta_{i-1}} < b < 0$, for all i.

Similarly, if $\gamma(t)=\theta_{i+1}$, for $\theta\le t<\theta_{i+1}$, then $Q=\prod_{i=p}^{1}[1+b(\theta_{i+1}-\theta_i)]^{-1}$, and the equation is uniformly exponentially stable, provided that $b>0$ or $b<\frac{-2}{\theta_i-\theta_{i-1}}$, for all i. Thus, we obtain that the simple Malthus model admits decaying solutions, even with positive coefficient, if the piecewise constant argument with anticipation is inserted. This fact certainly requests a biological conceive. Let us finalize the study with exact estimations. Assume that $|Q|<1$, and denote $-\alpha=\frac{1}{\omega}\ln|Q|$. Moreover, set $R=\max_i\{\max\{\max_t|M_i(t)|,\max_t|M_i^{-1}(t)|\}\}$. Then, $|x(t,t_0,x_0)|\le R\mathrm{e}^{-\alpha(t-t_)}|x_0|$, for an arbitrary solution $x(t,t_0,x_0)$ of the equation.

Example 3. Consider the following system

$$z'(t)=\begin{pmatrix}0&1\\0&0\end{pmatrix}z+\begin{pmatrix}0&0\\q&0\end{pmatrix}z([t]). \tag{10}$$

We can see that $\theta_i=i,\zeta_i=\theta_{i+1}=i+1,i\in\mathbb{Z}$, for this system. One can find that

$$Q=G_i\equiv M_i^{-1}(\theta_i)M_i(\theta_{i+1})=\begin{pmatrix}1+q/2&1\\q&1\end{pmatrix}.$$

Denote by $\rho_j,\ j=1,2$, eigenvalues of matrix Q. They are *multipliers* of system (10). We find that

$$\rho_{1,2}=\frac{1}{1-q/2}[1+\frac{q}{4}\pm\sqrt{\frac{q^2}{16}+q}].$$

From (5), it implies that the zero solution of (10) is exponentially stable, if and only if absolute values of both multiplies less than one. We find that this is valid if $-16<q<0$. Moreover, if $0<q<2$, then the system is exponentially dichotomous, such that and $|\rho_1|>1$ and $|\rho_2|<1$.

3 The Non-homogeneous Linear System

Theorem 2. *Assume that all conditions stated above hold. Then the solution* $z(t)=z(t,t_0,z_0),(t_0,z_0)\in\mathbb{R}\times\mathbb{R}^n$, *of (2) is unique, defined on* $\mathbb{R}$, *and equal to*

$$\begin{aligned}z(t)=&Z(t,t_0)z_0+Z(t,t_0)\int_{t_0}^{\zeta_i}X(t_0,s)f(s)\,ds+\\&\sum_{k=i}^{k=j-1}Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}X(\theta_{k+1},s)f(s)\,ds+\int_{\zeta_j}^{t}X(t,s)f(s)\,ds,\end{aligned} \tag{11}$$

where $\theta_i\le t_0\le\theta_{i+1}$ *and* $\theta_j\le t\le\theta_{j+1},\ j>i$, *and*

$$\begin{aligned}z(t)=&Z(t,t_0)z_0+Z(t,t_0)\int_{t_0}^{\zeta_i}X(t_0,s)f(s)\,ds+\\&\sum_{k=i}^{k=j}Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}X(\theta_{k+1},s)f(s)\,ds+\int_{\zeta_j}^{t}X(t,s)f(s)\,ds,\end{aligned} \tag{12}$$

where $\theta_i\le t_0\le\theta_{i+1}$ *and* $\theta_j\le t\le\theta_{j+1},\ j<i$.

Proof. Consider only the increasing time case, since the opposite direction is very similar. It is sufficient to verify that the expression (11) satisfies the system, since the uniqueness is obvious. Another thing to check is the continuity of solutions. So, let us differentiate the formula. We have that

$$z'(t) = [Z(t,t_0)z_0 + Z(t,t_0)\int_{t_0}^{\zeta_i} X(t_0,s)f(s)\,ds + \sum_{k=i}^{k=j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s)\,ds+$$

$$\int_{\zeta_j}^{t} X(t,s)f(s)\,ds]' = [A_0(t)Z(t,t_0)+A_1(t)Z(\gamma(t),t_0)]z_0 + [A_0(t)Z(t,t_0)+A_1(t)Z(\gamma(t),t_0)]\times$$

$$\int_{t_0}^{\zeta_i} X(t_0,s)f(s)\,ds + \sum_{k=i}^{k=j-1}[A_0(t)Z(t,\theta_{k+1})+A_1(t)Z(\gamma(t),\theta_{k+1})]\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s)\,ds+$$

$$\int_{\zeta_j}^{t} A_0(t)X(t,s)f(s)\,ds + f(t) = A_0(t)z(t)+A_1(t)z(\gamma(t)) + f(t).$$

It is clear that $z(t)$ is continuous in each interval $(\theta_i,\theta_{i+1}), i\in\mathbb{Z}$. Then we have that for a fixed $j\in\mathbb{Z}$,

$$z(\theta_j+) = Z(\theta_j,t_0)z_0 + Z(\theta_j,t_0)\int_{t_0}^{\zeta_i} X(t_0,s)f(s)\,ds+$$

$$\sum_{k=i}^{k=j-1} Z(\theta_j,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s)\,ds + \int_{\zeta_j}^{\theta_j} X(\theta_j,s)f(s)\,ds$$

and

$$z(\theta_j-) = Z(\theta_j,t_0)z_0 + Z(\theta_j,t_0)\int_{t_0}^{\zeta_i} X(t_0,s)f(s)\,ds+$$

$$\sum_{k=i}^{k=j-2} Z(\theta_j,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s)\,ds + \int_{\zeta_{j-1}}^{\theta_j} X(\theta_j,s)f(s)\,ds$$

Subtract from the first formula the second to obtain that

$$z(\theta_j+) - z(\theta_j-) = Z(\theta_j,\theta_j)\int_{\zeta_{j-1}}^{\zeta_j} X(\theta_{k+1},s)f(s)\,ds+$$

$$\int_{\zeta_j}^{\theta_j} X(\theta_j,s)f(s)\,ds - \int_{\zeta_{j-1}}^{\theta_j} X(\theta_j,s)f(s)\,ds = 0.$$

The theorem is proved. □

The last theorem for quasilinear systems is proved in [24, 27].

Denote by $[\widehat{a,b}], a,b\in\mathbb{Z}$, the interval $[a,b]$, whenever $a\le b$, and $[b,a]$, otherwise. To see better similarity of our results with those for ordinary differential equations, denote by $\Phi(t,s)$ the piecewise matrix, which is defined in the following way,

$$\Phi(t,s) = \begin{cases} Z(\theta_j,t_0)X(t_0,s), & t\in[\widehat{t_0,\zeta_j}],\\ Z(t,\theta_{k+1})X(\theta_{k+1},s), & t\in[\zeta_k,\zeta_{k+1}],\\ X(t,s), & t\in[\widehat{\zeta_j,t}], \end{cases} \tag{13}$$

where $t_0 \in [\theta_i, \theta_{i+1}], t \in [\theta_j, \theta_{j+1}]$. Then, we can see that formulas (11) and (12) are unified as

$$z(t) = Z(t,t_0)z_0 + \int_{t_0}^{t} \Phi(t,s) f(s) ds. \tag{14}$$

To arrange a similarity with ordinary differential equations, we call the matrix $\Phi(t,s)$, the *Cauchy matrix*, and (14), the *Cauchy representation formula*. We have to make the following important remark for the definition of the Cauchy matrix, (13), and other piecewise matrices, which will be provided below, in the paper. We define the matrix continuous on *closed* intervals, which overlap at their ends, but this provides no difficulties for evaluation of solutions. Another type of integral representation of solutions can be found in paper [39].

Next, assume that matrices $A_j, j = 0,1$, are uniformly bounded on $\mathbb{R}$, there exists a number $\bar{\theta} > 0$ such that $\theta_{i+1} - \theta_i \le \bar{\theta}, i \in \mathbb{Z}$, and there exists a number $\underline{\theta} > 0$ such that $\theta_{i+1} - \theta_i \ge \underline{\theta}, i \in \mathbb{Z}$. One can easily see that above stated conditions imply existence of positive numbers M, m and $\bar{M}$ such that $m \le ||Z(t,s)|| \le M, \|X(t,s)\| \le \bar{M}$ if $t,s \in [\theta_i, \theta_{i+1}], i \in \mathbb{Z}$. Moreover, set $\tilde{M} = \sup_{\mathbb{R}} \|f\|$. Assume that the system (1) is exponentially dichotomous and write, in what follows, $Z_-(t,s) = Z(t)PZ^{-1}(s), Z_+ = Z(t)(I-P)Z^{-1}(s)$.

Theorem 3. *Assume that all conditions stated above hold, and the linear system (1) is exponentially dichotomous, then there is a unique bounded on $\mathbb{R}$ solution of (2),*

$$z(t) = \sum_{k=-\infty}^{j} Z_-(t,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s) f(s)\, ds - \sum_{k=j+1}^{\infty} Z_+(t,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s) f(s)\, ds + \int_{\zeta_j}^{t} X(t,s) f(s)\, ds, \tag{15}$$

Proof. It is an easy job to check that the expression satisfies the system, if series are convergent. So, let us verify that they do. Consider just the first one, since for another this can be done very similarly. We have that,

$$\| \sum_{k=-\infty}^{j} Z_-(t,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s) f(s)\, ds \| \le 2\bar{\theta}\bar{M}\tilde{M}R \frac{e^{\alpha\bar{\theta}}}{1 - e^{-\alpha\underline{\theta}}}.$$

Next, the function has to be continuous on $\mathbb{R}$. It is obvious that it is continuous in any interval $(\theta_i, \theta_{i+1}), i \in \mathbb{Z}$. So, only points $\theta_i, i \in \mathbb{Z}$, are suspicious.

Fix $j \in \mathbb{Z}$, and evaluate

$$z(\theta_j+) = \sum_{k=-\infty}^{j} Z_-(\theta_j,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s) f(s)\, ds - \sum_{k=j+1}^{\infty} Z_+(\theta_j,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s) f(s)\, ds + \int_{\zeta_j}^{\theta_j} X(t,s) f(s)\, ds$$

and

$$z(\theta_j-) = \sum_{k=-\infty}^{j-1} Z_-(\theta_j,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s) f(s)\, ds - \sum_{k=j}^{\infty} Z_+(\theta_j,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s) f(s)\, ds + \int_{\zeta_{j-1}}^{\theta_j} X(t,s) f(s)\, ds$$

Subtract from the first formula the second one to see that $z(\theta_j+) = z(\theta_j-) = 0$.

The theorem is proved. □

One can easily see that the bounded solution, whose existence is proved by the last theorem, is uniformly asymptotically stable, if there exist constants, $R \geq 1, \alpha > 0$, such that $\|Z(t,s)\| \leq R\mathrm{e}^{-\alpha(t-s)}, s \leq t$.

To provide more compact formula for future investigations, let us introduce the following notations.

$$\Phi_-(t,s) = \begin{cases} Z_-(t,\theta_{k+1})X(\theta_{k+1},s), & s \in [\zeta_k, \zeta_{k+1}], \\ X(t)PX^{-1}(s), & s \in [\hat{\zeta_j}, t], \end{cases} \tag{16}$$

and

$$\Phi_+(t,s) = \begin{cases} Z_+(t,\theta_{k+1})X(\theta_{k+1},s), & s \in [\zeta_k, \zeta_{k+1}], \\ X(t)(I-P)X^{-1}(s), & s \in [\hat{\zeta_j}, t], \end{cases} \tag{17}$$

where $t \in [\theta_j, \theta_{j+1}]$. Then, we can see that formula (15) can be written as

$$z(t) = \int_{-\infty}^{t} \Phi_-(t,s)f(s)ds + \int_{t}^{\infty} \Phi_+(t,s)f(s)ds. \tag{18}$$

Next, if we write

$$G_B(t,s) = \begin{cases} \Phi_+(t,s), & s \leq t, \\ \Phi_-(t,s), & t < s, \end{cases}$$

(18) can be presented as

$$z(t) = \int_{-\infty}^{\infty} G_B(t,s)f(s)ds, \tag{19}$$

where $G_B(t,s)$ is the *Green function* for a *bounded solution*.

Example 4. Consider the following system

$$z'(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} z(\gamma(t)) + f(t). \tag{20}$$

Where $\theta_i = i + \delta_i, \zeta_i = \theta_i, i \in \mathbb{Z}$. Assume that $|\delta_i| < \delta$, where δ is a positive parameter. Use the result of Example 3, to obtain that, if $q \neq -\frac{4}{3}, 4$, then the system (20) is exponentially dichotomous. Indeed, it is seen immediately through formula for $M_i(t)$, that the eigenvalues continuously depend on the parameter, δ, and results from [42]. Assume, now, that function $f(t)$ is a bounded one, then from the last theorem it implies that there exists a unique bounded on $\mathbb{R}$ solution of (20).

4 Periodic solutions

Assume that there are two numbers, $\omega \in \mathbb{R}, p \in \mathbb{Z}$, such that $\theta_{k+p} = \theta_k + \omega, \zeta_{k+p} = \zeta_k + \omega, k \in \mathbb{Z}$. Then denote by Q the product $\prod_{k=1}^{p} G_k$, where matrices G_k are equal to $M_k^{-1}(\theta_k)M_{k-1}(\theta_k), k \in \mathbb{Z}$. The matrix Q, is the monodromy matrix, and eigenvalues of the matrix, $\rho_j, j = 1,2,\ldots,n$, are multipliers. It is clear that system (1) admits a periodic solution, if there exists a unit multiplier. Generally, all the results known for linear homogeneous ordinary differential equations based on the unit multipliers can be identically repeated for the present systems. Our main goal in this section is to study the non-critical systems, and find formulas for solutions. We assume that the system is $\omega-$ periodic. That is, in addition to the above conditions, $f(t+\omega) = f(t), t \in \mathbb{R}$. In what follows, we assume without loss of generality that $\zeta_0 = 0$, and consider $t_0 = \zeta_0$.

Consider the solution $z(t) = z(t,0,z_0)$. We have that

$$z(t) = Z(t,0)z_0 + \sum_{k=0}^{k=j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds + \int_{\zeta_j}^{t} X(t,s) f(s)\, ds, \tag{21}$$

and

$$z(\omega) = Z(\omega,0)z_0 + \sum_{k=0}^{k=p-1} Z(\omega,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds.$$

In papers [35, 36] we proved the Poincar*é* criterion for EPCAG. According to this, $z(t)$ is a periodic solution if and only if z_0 satisfies

$$[I - Z(\omega,0)]z_0 = \sum_{k=0}^{k=p-1} Z(\omega,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds.$$

By conditions of non-criticality, $\det[I - Z(\omega,0)] \neq 0$, and the last equation admits a unique solution,

$$z^* = [I - Z(\omega,0)]^{-1} \sum_{k=0}^{k=p-1} Z(\omega,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds.$$

Thus, we have obtained that

$$z(t) = Z(t,0)[I - Z(\omega,0)]^{-1} \sum_{k=0}^{p-1} Z(\omega,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds + \sum_{k=0}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds + \int_{\zeta_j}^{t} X(t,s) f(s)\, ds. \tag{22}$$

Use formula (22) to obtain

$$z(t) = \sum_{k=0}^{k=j-1} Z(t)[I - Z(\omega)]^{-1} Z^{-1}(\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds + \sum_{k=j}^{k=p-1} Z(t)[I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s)\, ds + \int_{\zeta_j}^{t} X(t,s) f(s)\, ds. \tag{23}$$

One can easily verify by substitution that (23) is a solution , and it is a continuous function. One can construct the following *Green function* for the *periodic solution*, $G_P(t,s), t,s \in [0,\omega]$.

If $t \in [\theta_j, \theta_{j+1}), j = 0,2,\ldots,p-1$, then

$$G_P(t,s) = \begin{cases} Z(t)[I-Z(\omega)]^{-1}Z^{-1}(\theta_{k+1})X(\theta_{k+1},s), & s \in [\zeta_k, \zeta_{k+1}), k < j, \\ Z(t)[I-Z(\omega)]^{-1}Z(\omega)Z^{-1}(\theta_{k+1})X(\theta_{k+1},s), & s \in [\zeta_k, \zeta_{k+1}) \setminus [\widehat{\zeta_j,t}], k \geq j, \\ Z(t)[I-Z(\omega)]^{-1}Z(\omega)Z^{-1}(\theta_{k+1})X(\theta_{k+1},s) + X(t,s), & s \in [\widehat{\zeta_j,t}] \end{cases}$$

Now, apply the last formula in (23) to see that the periodic solution can be written as

$$z(t) = \int_0^\omega G_P(t,s) f(s) ds.$$

Finally, the Green function for a bounded solution, $G_B(t,s)$, in this periodic case satisfies $G_B(t+\omega, s+\omega) = G_P(t,s), t,s \in \mathbb{Z}$, and consequently, the periodic solution can be written as (19).

5 Almost periodic solutions

From now on we make the assumption that $\|Z(t,s)\| \leq R\mathrm{e}^{-\alpha(t-s)}$, for some numbers $R \geq 1, \alpha > 0$. Let $C_0(\mathbb{R})$ be the set of all bounded and uniformly continuous on $\mathbb{R}$ functions. For $f \in C_0(\mathbb{R})$ and $\tau \in \mathbb{R}$ the translate of f by τ is the function $Q_\tau f = f(t+\tau), t \in \mathbb{R}$. A number $\tau \in \mathbb{R}$ is called $\varepsilon-$ translation number of a function $f \in C_0(\mathbb{R})$ if $||Q_\tau f - f|| < \varepsilon$ for every $t \in \mathbb{R}$. A function $f \in C_0(\mathbb{R})$ is called almost periodic if for every $\varepsilon > 0$, there exists a respectively dense set of $\varepsilon-$ translations of f, [45–47]. Denote by $\mathscr{A}\mathscr{P}(\mathbb{R})$ the set of all almost periodic functions. Let $\zeta_i^j = \zeta_{i+j} - \zeta_i, \theta_i^j = \theta_{i+j} - \theta_i$ for all i and j. We call the family of sequences $\{\zeta_i^j\}_i, j \in \mathbb{Z}$, equipotentially almost periodic [44, 48] if for an arbitrary positive ε there exists a relatively dense set of $\varepsilon-$ translation numbers, common for all sequences $\{\zeta_i^j\}_i, j \in \mathbb{Z}$. In what follows we assume that sequences $\zeta_i^j, j \in \mathbb{Z}$, as well as sequences $\theta_i^j, j \in \mathbb{Z}$, are equipotentially almost periodic. This condition implies that $|\theta_i|, |\zeta_i| \to \infty$, as $|i| \to \infty$. Moreover, it follows [44, 48], that there exist positive numbers $\overline{\theta}$ and $\overline{\zeta}$ such that $\theta_{i+1} - \theta_i \leq \overline{\theta}, \zeta_{i+1} - \zeta_i \leq \overline{\zeta}, i \in \mathbb{Z}$. Additionally, we assume that there are two positive numbers, $\underline{\zeta}, \underline{\theta}$, such that $\theta_{i+1} - \theta_i \geq \underline{\theta}, \zeta_{i+1} - \zeta_i \geq \underline{\zeta}, i \in \mathbb{Z}$.

The following assertion is a specific one. It connects almost periodicity not only of functions, but functions and sequences. The technique of its proof was developed in papers of D. Wexler [43]. See, for example, [44, 48], where functions and sequences were discussed. Later, the technique was effectively used for discontinuous almost periodic functions and sequences in [34]. See, also, [48]. One can easily extend the results for almost periodic functionals and sequences.

Lemma 4. *[44, 48] Assume that functions $\phi_j(t), j = 1,2,\ldots,k$, are almost periodic in t, $\theta_i^j, \zeta_i^j, j \in \mathbb{Z}$, are equipotentially almost periodic and $\inf_{\mathbb{Z}} \theta_i^1, \inf_{\mathbb{Z}} \zeta_i^1 > 0$. Then, for arbitrary $\eta > 0, 0 < \nu < \eta$, there exist a respectively dense set of real numbers Ω and integers Q, such that for $\omega \in \Omega, q \in Q$, it is true that*

1. $\|\phi_j(t+\omega) - \phi_j(t)\| < \eta, j = 1,2,\ldots,k, t \in \mathbb{R}$;
2. $|\theta_i^q - \omega| < \nu, i \in \mathbb{Z}$;
3. $|\zeta_i^q - \omega| < \nu, i \in \mathbb{Z}$.

Let us prove an auxiliary assertion.

Lemma 5. *Let $\omega \in \mathbb{R}$ be a common $\eta-$ almost period of matrices $A_0(t), A_1(t)$, then*

$$\|Z(t+\omega,s+\omega)-Z(t+\omega,s+\omega)\| < \frac{R\bar{M}e^{\alpha\bar{\theta}}}{\alpha}\eta e^{-\frac{\alpha}{2}(t-s)}, s \le t. \tag{24}$$

Proof. Set $W(t,s) = Z(t+\omega,s+\omega)-Z(t,s)$. Then

$$\frac{\partial W}{\partial t} = A_0(t)W(t,s)+A_1(t)W(\gamma(t),s)+[A_0(t+\omega)-A_0(t)]Z(t+\omega,s+\omega)+$$
$$[A_1(t+\omega)-A_0(t)]Z(\gamma(t)+\omega,s+\omega).$$

Since $W(s,s) = 0$, from the last equation it follows that

$$W(t,s) = Z(t,s)\int_s^{\zeta_i} X(s,u)[A_0(u+\omega)-A_0(u)]Z(u+\omega,s+\omega)+$$
$$[A_1(u+\omega)-A_1(u)]Z(\gamma(u)+\omega,s+\omega)]\,du+$$
$$\sum_{k=i}^{k=j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},u)[A_0(u+\omega)-A_0(u)]Z(u+\omega,s+\omega)+$$
$$[A_1(u+\omega)-A_1(u)]Z(\gamma(u)+\omega,s+\omega)]\,du+$$
$$\int_{\zeta_j}^t X(t,u)[A_0(u+\omega)-A_0(u)]Z(u+\omega,s+\omega)+[A_1(u+\omega)-A_1(u)]Z(\gamma(u)+\omega,s+\omega)]\,du.$$

Then, we have that

$$\|W(t,s)\| \le \int_s^{\zeta_i} R\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du+\sum_{k=i}^{k=j-1}\int_{\zeta_k}^{\zeta_{k+1}} R\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du+\int_{\zeta_j}^t R\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du \le$$
$$\int_s^t R\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du = R\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}(t-s) \le \frac{R\bar{M}e^{\alpha\bar{\theta}}}{\alpha}\eta e^{-\frac{\alpha}{2}(t-s)}.$$

The lemma is proved. □

Theorem 6. *Suppose that all conditions stated above hold. If $f \in \mathscr{AP}(\mathbb{R})$, then equation (2) admits a unique exponentially stable almost periodic solution.*

Proof. Consider formula (15). Since the projection $P = I$, this time the expression admits the form,

$$z(t) = \sum_{k=-\infty}^{j} Z(t,\theta_k)\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds+\int_{\zeta_j}^t X(t,s)f(s)\,ds. \tag{25}$$

Let us check that the bounded solution is almost periodic. Take ω and q that satisfy Lemma 4 with functions A_0, A_1, f. Then, there exist numbers $\nu_1,\nu_2,|\nu_j| < \eta$, $j = 1,2$, such that $\zeta_{k+q+1} = \zeta_{k+1}+\omega+\nu_2$ and $\zeta_{k+q} = \zeta_k+\omega+\nu_1$. Then,

$$z(t+\omega)-z(t) = \sum_{k=-\infty}^{j+q} Z(t+\omega,\theta_k)\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds+\int_{\zeta_{j+q}}^{t+\omega} X(t+\omega,s)f(s)\,ds-$$

$$\sum_{k=-\infty}^{j} Z(t,\theta_k)\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds + \int_{\zeta_j}^{t} X(t,s)f(s)\,ds =$$

$$\sum_{k=-\infty}^{j} Z(t+\omega,\theta_{k+q})\int_{\zeta_{k+q-1}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\,ds + \int_{\zeta_j}^{t} X(t+\omega,s+\omega)f(s+\omega)\,ds -$$

$$\sum_{k=-\infty}^{j} Z(t,\theta_k)\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds + \int_{\zeta_j}^{t} X(t,s)f(s)\,ds =$$

$$\sum_{k=-\infty}^{j} [Z(t+\omega,\theta_{k+q})\int_{\zeta_{k-1+q}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\,ds - Z(t,\theta_k)\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds +$$

$$\int_{\zeta_{j+q}}^{t+\omega} X(t+\omega,s)f(s)\,ds - \int_{\zeta_j}^{t} X(t,s)f(s)\,ds.$$

We have that

$$\|Z(t+\omega,\theta_{k+q}) - Z(t,\theta_k)\| \le \|Z(t+\omega,\theta_{k+q}) - Z(t+\omega,\theta_k+\omega)\| +$$
$$\|Z(t+\omega,\theta_k+\omega) - Z(t,\theta_k)\|.$$

Then

$$\|Z(t+\omega,\theta_{k+q}) - Z(t+\omega,\theta_k+\omega)\| \le \|Z(t+\omega,\theta_{k+q})\|\|I - Z(\theta_{k+q},\theta_k+\omega)\| \le$$
$$KR_1(\eta)\mathrm{e}^{-\alpha(t+\omega-\theta_{k+1+q})},$$

where $R_1 \to 0$ as $\eta \to 0$. Moreover,

$$\|Z(t+\omega,\theta_k+\omega) - Z(t,\theta_k)\| \le \frac{R\bar{M}\mathrm{e}^{\alpha\bar{\theta}}}{\alpha}\eta\,\mathrm{e}^{-\frac{\alpha}{2}(t-\theta_k)},$$

according to Lemma 5.

Thus, it is true that

$$\|Z(t+\omega,\theta_{k+q}) - Z(t,\theta_k)\| \le R_2(\eta)\mathrm{e}^{-\frac{\alpha}{2}(t-\theta_k)},$$

where $R_2(\eta)$ is a positive valued function such that $R_2(\eta) \to 0$, as $\eta \to 0$.

Let us make the following transformations,

$$\int_{\zeta_{k+q-1}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\,ds - \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds =$$

$$\int_{\zeta_{k-1}+\omega+\nu_1}^{\zeta_{k-1}+\omega} X(\theta_{k+q},s)f(s)\,ds + \int_{\zeta_k+\omega+\nu_2}^{\zeta_k+\omega} X(\theta_{k+q},s)f(s)\,ds +$$

$$\int_{\zeta_{k-1}}^{\zeta_k} [X(\theta_{k+q},s+\omega)f(s+\omega) - X(\theta_k,s)f(s)\,ds$$

Apply to the last expression a discussion that is similar to the one made above to achieve that

$$\|\int_{\zeta_{k+q-1}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\,ds - \int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds\| \le R_3(\eta),$$

where $R_3 \to 0$, as $\eta \to 0$.
Then, we have that

$$\|Z(t+\omega,\theta_{k+q})\int_{\zeta_{k-1+q}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\;ds - Z(t,\theta_k)\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\;ds\| \le$$

$$\|Z(t+\omega,\theta_{k+q}) - Z(t,\theta_k)\|\|\int_{\zeta_{k+q-1}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\,ds\| + \|\int_{\zeta_{k+q-1}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\,ds -$$

$$\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)\,ds\|\|Z(t,\theta_k)\| \le \bar{\zeta}\bar{M}\tilde{M}R_2(\eta)\mathrm{e}^{-\frac{\alpha}{2}(t-\theta_k)} + R_3(\eta)\mathrm{e}^{-\alpha(t-\theta_k)} = R_4(\eta)\mathrm{e}^{-\frac{\alpha}{2}(t-\theta_k)}.$$

Hence,

$$\|\sum_{k=-\infty}^{j}[Z(t+\omega,\theta_{k+q})\int_{\zeta_{k-1+q}}^{\zeta_{k+q}} X(\theta_{k+q},s)f(s)\;ds - Z(t,\theta_k)\int_{\zeta_{k-1}}^{\zeta_k} X(\theta_k,s)f(s)]\;ds\| \le$$

$$\sum_{k=-\infty}^{j} R_4(\eta)\mathrm{e}^{-\frac{\alpha}{2}(t-\theta_k)} = R_4(\eta)\frac{\mathrm{e}^{\frac{\alpha}{2}\bar{\theta}}}{1-\mathrm{e}^{-\frac{\alpha}{2}\underline{\theta}}}.$$

Now, use the "diagonal almost periodicity" of $X(t,s)$, [44], and almost periodicity of f, to attain that

$$\|X(t+\omega,s+\omega)f(s+\omega) - X(t,s)f(s)\| \le \|X(t+\omega,s+\omega) - X(t,s)\|\|f(s+\omega)\| +$$

$$\|f(s+\omega) - f(s)\|\|X(t,s)\| \le R_5(\eta)\tilde{M} + \eta\bar{M} = R_6(\eta).$$

where $R_6(\eta) \to 0$ as $\eta \to 0$. Next, we have that

$$\|\int_{\zeta_{j+q}}^{t+\omega} X(t+\omega,s)f(s)\;ds - \int_{\zeta_j}^{t} X(t,s)f(s)\;ds\| \le \|\int_{\zeta_{j+\omega+\nu_1}}^{\zeta_j+\omega} X(t+\omega,s)f(s)\;ds\| +$$

$$\|\int_{\zeta_j}^{t}[X(t+\omega,s+\omega)f(s+\omega) - X(t,s)f(s)]\;ds\| \le \bar{M}\tilde{M}\eta + R_6(\eta)\bar{\theta} = R_7(\eta).$$

Apply the last estimations, to obtain that

$$\|z(t+\omega) - z(t)\| \le R_4(\eta)\frac{\mathrm{e}^{\frac{\alpha}{2}\bar{\theta}}}{1-\mathrm{e}^{-\frac{\alpha}{2}\underline{\theta}}} + R_7(\eta) = R_8(\eta),$$

where $R_8 \to 0$ as $\eta \to 0$. The last inequality proves the almost periodicity of $z(t)$. Stability of the solution as well as its uniqueness , follows immediately those for the homogeneous system.

The theorem is proved. □

Example 5. In [34] (see, also, [48]), we proved that the sequence $\theta_i = i + a_i$, where $a_i = \frac{1}{4}|\sin(i) - \cos(i\sqrt{2})|$, satisfies conditions of the last theorem. That is, sequences θ_i^j are equipotentially almost periodic, and there are positive numbers $\overline{\theta}$ and $\underline{\theta}$ such that $\underline{\theta} < \theta_{i+1} - \theta_i \le \overline{\theta}, i \in \mathbb{Z}$. Now, introduce a sequence $\{\zeta_i\}$ such that $\zeta_i = \theta_{i+1}$. It is obvious that $\overline{\zeta} = \overline{\theta}, \underline{\zeta} = \underline{\theta}$. Introduce the following EPCAG,

$$z'(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} z(\gamma(t)) + f(t). \tag{26}$$

Write $\pi_i = \theta_{i+1} - \theta_i$, and find that

$$M_i^{-1}(\theta_i)M_i(\theta_{i+1}) = \frac{1}{1 - \frac{q}{2}\pi_i^2} \begin{pmatrix} 1 & \pi_i^2 \\ q\pi_i^2 & 1 + \frac{q}{2}\pi_i^2 \end{pmatrix}.$$

Then,

$$\|M_i^{-1}(\theta_i)M_i(\theta_{i+1})\| < 1,$$

if $-2 - \sqrt{3 - \frac{2}{\pi_i^2}} < q < -2 + \sqrt{3 - \frac{2}{\pi_i^2}}$.

Since $\sup_i \pi_i^2 = 1 + \sup_i (a_{i+1} - a_i) \le \frac{3}{2}$, we finally obtain that there exists a unique exponentially stable almost periodic solution of (26) if $-2 - \sqrt{5/3} < q < -2 + \sqrt{5/3}$. Now, let us change the conditions for (26). Assume that $\zeta_i \ne \theta_{i+1}$, but $\theta_{i+1} - \zeta_i \le \varepsilon \le \underline{\theta}$, where ε is a positive parameter. For example, one can take $\zeta_i = \theta_{i+1} - \varepsilon$. From continuous dependence of solutions and norms of matrices on parameters it follows, that (26) admits a unique exponentially stable almost periodic solution if $-2 - \sqrt{5/3} < q < -2 + \sqrt{5/3}$, and ε is sufficiently small.

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