METU, Spring 2014, Math 504.

Homework 6

(due May 21)

1. Determine the splitting field and its degree over $\mathbf{Q}$ for $f(x) = x^6 + 3$.

2. If F is algebraically closed and E consists of all elements in F that are algebraic over K, then show that E is an algebraic closure of K.

3. Prove that an algebraic extension F of K is normal over K if and only if for every irreducible $f \in K[x]$, f factors in $F[x]$ as a product of irreducible factors al of which have the same degree.

4. Let $f(x) = x^3 + bx^2 + cx + d$ be a cubic irreducible polynomial over $\mathbf{Q}$. Let G be the Galois group of the polynomial f over $\mathbf{Q}$. Are the following true? Prove or disprove.

   - If roots of f are real then $G \cong A_3$.
   - If $G \cong A_3$, then roots of f must be real.

5. Let $f(x) = x^4 - x^3 + 3x^2 + x + 1$. Show that f is irreducible over $\mathbf{Q}$. Let F be its splitting field. Determine the Galois group of $F/\mathbf{Q}$. Show that $E = \mathbf{Q}(\sqrt{5}) \subset F$ and determine the corresponding subgroup E' of $\mathrm{Aut}_K F$.