METU, Spring 2014, Math 504.

Homework 4

(due April 9)

1. Let A be cyclic module of order $r \in R$. Prove the following:
   - If $s \in R$ is relatively prime to r, then $sA = A$ and $A[s] = 0$.
   - If s divides r, say $sk = r$, then $sA \cong R/(k)$ and $A[s] = R/(s)$.
2. The annihilator of an R-module A is defined by $\mathrm{Ann}(A) = \{r \in R : rA = 0\}$.
   - If A is a finitely generated torsion module then show that $\mathrm{Ann}(A)$ is a nonzero ideal in R.
   - Give an example of an integral domain R and a nonzero torsion R-module A so that $\mathrm{Ann}(A) = 0$.
   - Let A be a finite abelian group with $\mathrm{Ann}(A) = (m)$ for some $m \in \mathbf{Z}$. Show that a cyclic subgroup of A of order properly dividing m need not be a direct summand of A.
3. If $T : V \to V$ is a linear transformation on a vector space V over the field $\mathbf{R}$, then the vector space V can be made into an $\mathbf{R}[x]$-module by setting $x \cdot v = T(v)$ for any $v \in V$. For each of the following transformations T, find a decomposition of V according to the Theorem 6.12 in our textbook. Provide both type of decompositions.
   - $V = \mathbf{R}^2$ and $T(x, y) = (-y, x)$.
   - $V = \mathbf{R}^2$ and $T(x, y) = (0, y)$.
   - $V = \mathbf{R}^3$ and $T(x, y, z) = (z, x, y)$.
4. Let $R = \mathbf{Z}[i]$ be the ring of Gaussian integers. Let $\alpha = 30$ and $\beta = (2 + i)(1 + i)$ be elements of R generating the ideals $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Consider the R-module

   $$A = R/\mathfrak{a} \oplus R/\mathfrak{b}.$$

   Decompose the R-module A according to the Theorem 6.12 in our textbook. Provide both type of decompositions.