METU, Spring 2014, Math 504.

Homework 2

(due March 12)

1. Let $\{x_1, x_2, x_3\}$ be a linearly independent subset of a vector space V.

   - Show that $\{x_1 + x_2 + x_3, x_2 + x_3, x_3\}$ is a linearly independent subset of V.
   - Is it necessarily true that the subset $X = \{x_1 + x_2, x_2 + x_3, x_3 + x_1\}$ of V is linearly independent? If not, then find a sufficient condition so that X is linearly independent.

2. Prove that the space of real-valued functions on the closed interval $[0, 1]$ is an infinite dimensional vector space over $\mathbf{R}$. Is its subset consisting of continuous functions a subspace?

3. Let $f : V \to V'$ be a linear transformation of finite dimensional vector spaces where $\dim(V) = \dim(V')$. Show that the following conditions are equivalent:

   - f is an isomorphism,
   - f is an epimorphism,
   - f is a monomorphism.

4. Let φ be a linear transformation of the finite dimensional vector space V to itself such that $\varphi^2 = \varphi$.

   - Prove that $\mathrm{Im}(\varphi) \cap \mathrm{Ker}(\varphi) = 0$.
   - Prove that $V = \mathrm{Im}(\varphi) \oplus \mathrm{Ker}(\varphi)$.

5. Let V be a finite dimensional vector space with basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ over a field F. Let $V^* = \mathrm{Hom}_F(V, F)$ be the space of linear transformations from V to F. Define $v_i^* \in V^*$ for each $i \in \{1, 2, \ldots, n\}$ as follows:

$$v_i^*(v_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

   Show that $\{v_1^*, v_2^*, \ldots, v_n^*\}$ is a basis for V^*.