METU, Spring 2014, Math 504.

Homework 1

(due March 5)

1. If M is a finite abelian group then it is naturally a $\mathbf{Z}$-module. Can this action be extended to make M into a $\mathbf{Q}$-module?

2. If $T : V \to V$ is a linear transformation on a vector space V over a field $\mathbf{F}$, then V can be made into an $\mathbf{F}[x]$-module by setting $xv = Tv$ for any $v \in V$. For each of the following transformations T, find all $\mathbf{F}[x]$-submodules of V.

   - $V = \mathbf{R}^2$ and $T(x, y) = (-y, x)$.
   - $V = \mathbf{R}^2$ and $T(x, y) = (0, y)$.
   - $V = \mathbf{R}^3$ and $T(x, y, z) = (z, x, y)$.

3. An element m of the R-module M is called a torsion element if $rm = 0$ for some non-zero $r \in R$. The set of torsion elements in M is denoted by $\mathrm{Tor}(M)$.

   - If R is an integral domain then prove that $\mathrm{Tor}(M)$ is a submodule of M. Give an example so that $\mathrm{Tor}(M)$ is not a submodule.
   - Show that if R has zero divisors than every non-zero R-module has torsion elements.

4. Give an example of a finitely generated R-module which is not finitely generated as an abelian group.

5. Suppose that

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & D \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\ A' & \to & B' & \to & C' & \to & D' \end{array}$$

   is a commutative diagram of modules and suppose that the rows are exact. Prove the following statements.

   - If α is surjective, and β, δ are injective, then γ is injective.
   - If δ is injective, and α, γ are surjective, then β is surjective.