M E T U
Department of Mathematics

Abstract Algebra Midterm 2	
Code : *Math 367* Acad. Year : *2015* Semester : *Fall* Instructor : *Küçüksakallı* Date : *Dec 14, 2015* Time : *17:40* Duration : *120 minutes*	Last Name : Name : Student No. : Signature : 6 QUESTIONS ON 4 PAGES 100 TOTAL POINTS

1	2	3	4	5	6		

1. (25pts) For each of the following statements determine if it is **true** or **false**. Explain your answer briefly.

- Let G be a finite group and p be a prime number. There exists an element $a \in G$ of order p if and only if p divides $|G|$.

- Let G be a finite group such that $|G|$ is divisible by p^2 where p is prime. Then there exists an element $a \in G$ of order p^2.

- Let G be a finite group such that $|G|$ is divisible by p^2 where p is prime. Then there exists a subgroup $H \leq G$ of order p^2.

- The set $S = \{2a + b\sqrt{367} \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{R}$.

- The subrings $2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\}$ and $3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\}$ of $\mathbb{Z}$ are isomorphic.

2a. (5pts) State the class equation.

Theorem(Class Equation): Let ________________________________. Then

$$|G| = \underline{\hspace{3cm}} + \sum \underline{\hspace{6cm}}$$

where ____.

2b. (10pts) If G is a finite p-group with $|G| > 1$, then show that $|Z(G)| > 1$.

3. (10pts) Let G be a group of order 105.

- Show that G is not simple.

- Show that G has a subgroup of order 35.

4. (25pts) Let $M_2(\mathbb{Q})$ be the ring of 2×2 matrices with rational entries under the usual matrix addition and multiplication. Consider

$$R = \left\{ \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] \,\middle|\, a, b, c \in \mathbb{Q} \right\} \quad \text{and} \quad I = \left\{ \left[\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right] \,\middle|\, b \in \mathbb{Q} \right\}.$$

- Show that R is a subring of $M_2(\mathbb{Q})$.

- Show that I is not an ideal of $M_2(\mathbb{Q})$.

- Show that I is an ideal of R.

- Show that the map $f\left(\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]\right) = (a, c)$ is a ring homomorphism from R to $\mathbb{Q} \times \mathbb{Q}$. (Here $\mathbb{Q} \times \mathbb{Q}$ is the usual ring with componentwise addition and multiplication.)

- Show that the quotient ring R/I is isomorphic to $\mathbb{Q} \times \mathbb{Q}$.

5. (15pts) Set $i = \sqrt{-1}$ and consider the subring $R = \{a + bi \mid a, b \in \mathbb{Z}\}$ of $\mathbb{C}$. Let I be the ideal of R generated by 2 and $3 + i$, i.e. $I = \langle 2, 3 + i \rangle$.

- Show that $I = \langle 1 + i \rangle$.

- Determine the number of elements in the quotient ring R/I.

6. (10pts) Show that any finite field has order p^n, where p is prime. (Hint: Use the fundamental theorem of finite Abelian groups.)