M E T U
Department of Mathematics

Abstract Algebra Midterm 1	
Code : *Math 367* Acad. Year : *2015* Semester : *Fall* Instructor : *Küçüksakallı* Date : *Nov 9, 2015* Time : *17:40* Duration : *100 minutes*	Last Name : Name : Student No. : Signature : 6 QUESTIONS ON 4 PAGES 100 TOTAL POINTS

1	2	3	4	5	6		

1. (24pts) Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 6 & 1 & 2 & 7 & 3 & 5 \end{pmatrix}$ and $\tau = (123)(345)(678)$ in S_8.

- Express σ as a product of transpositions.

  $\sigma = (18524)(367) = (14)(12)(15)(18)(37)(36)$.

- Express τ as a product of disjoint cycles.

  $\tau = (12345)(678)$.

- Find σ^{100}.

  We have $\sigma^{100} = [(18524)(367)]^{100} = (18524)^{100}(367)^{100} = (367)^{100} = (367)$. The second equality holds because disjoint cycles commute with each other.

- Is it possible to find a permutation $\gamma \in S_8$ such that $\gamma\sigma\gamma^{-1} = \tau$? If your answer is yes, then find such a permutation.

  The permutations σ and τ have the same cycle type thus they are conjugate to each other. A possible choice for γ is (8245367).

2. (24pts) Let $G = \{(x, y) \in \mathbb{R}^2 \mid x^2+4y^2 = 4\}$. Set $(x_1, y_1)\star(x_2, y_2) = \left(\frac{x_1x_2}{2} - 2y_1y_2, \frac{x_1y_2 + x_2y_1}{2}\right)$.

- Consider $P = (6/5, 4/5)$. Is P an element of G? Is $P \star P$ an element of G?

  Since $(6/5)^2 + 4(4/5)^2 = 4$, we have $P \in G$. Note that $P \star P = (-14/25, 24/25)$. It is easily verified that $(-14/25)^2 + 4(24/25)^2 = 4$. Thus $P \star P$ is an element of G as well.

- Show that $\star$ is a binary operation on G.

  The ellipse $x^2 + 4y^2 = 4$ can be parametrized by $(2\cos\alpha, \sin\alpha)$. Let $P_1, P_2 \in G$. Then $P_1 = (2\cos\alpha_1, \sin\alpha_1)$ and $P_2 = (2\cos\alpha_2, \sin\alpha_2)$ for some real numbers α_1 and α_2. The angle addition formulas for cosine and sine implies that

  $$P_1 \star P_2 = (2\cos(\alpha_1 + \alpha_2), \sin(\alpha_1 + \alpha_2)).$$

  This shows that the resulting element $P_1 \star P_2$ is in G as well.

- Show that $(G, \star)$ is a group.

  The set G is nonempty because $(6/5, 4/5) \in G$. Associativity of the binary operation $\star$ follows from the associativity of real numbers. The identity element is $e = (2\cos 0, \sin 0) = (2, 0)$. The inverse of an element $(x, y) = (2\cos\alpha, \sin\alpha)$ is found by

  $$(x, y)^{-1} = (2\cos(-\alpha), \sin(-\alpha)) = (2\cos\alpha, -\sin\alpha) = (x, -y).$$

  We conclude that G is a group under the binary operation $\star$.

- Let $H = G \cap \mathbb{Q}^2$. Show that $H \leq G$.

  The set H is nonempty because $(6/5, 4/5) \in H$. Pick $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ from H. The coordinates x_i and y_i are rational for each $i = 1, 2$. We have

  $$\begin{aligned} P_1 \star P_2^{-1} &= (x_1, y_1) \star (x_2, -y_2) \\ &= \left(\frac{x_1x_2}{2} + 2y_1y_2, \frac{-x_1y_2 + x_2y_1}{2}\right). \end{aligned}$$

  Note that the point $P_1 \star P_2^{-1}$ has rational coordinates and it belongs to H. We conclude that H is a subgroup of G.

3. (16pts) Let G be a group. Let H be the subgroup of G generated by the squares of elements in G, i.e. $H = \langle \{g^2 | g \in G\} \rangle$.

- Show that $H \trianglelefteq G$.

  Pick $h \in H$. Then $h = \prod_{i=1}^{n} g_i^2$ for some $g_i \in G$. Let g be an element of G. Then

  $$ghg^{-1} = g\left(\prod_{i=1}^{n} g_i^2\right) g^{-1} = \prod_{i=1}^{n} \left(g\left(g_i^2\right) g^{-1}\right) = \prod_{i=1}^{n} \left(gg_ig^{-1}\right)^2.$$

  It follows that $ghg^{-1} \in H$. Therefore $H \trianglelefteq G$.

- Show that G/H is commutative.

  Any nontrivial element $gH \in G/H$ has order two because $(gH)(gH) = g^2H = H$. Pick g_1H and g_2H from G/H. Then $((g_1H)(g_2H))^2 = (g_1H)(g_2H)(g_1H)(g_2H) = H$. Multipliying the last equality from each side by appropriate terms, we obtain that $(g_1H)(g_2H) = (g_2H)(g_1H)$. We conclude that G/H is commutative.

4. (10pts) Show that the groups $(\mathbb{C}, +)$ and $(\mathbb{C} - \{0\}, \times)$ are not isomorphic.

If $f : G \to H$ is an isomorphism then each element $g \in G$ of finite order is mapped to an element $h \in H$ with the same order. Note that the group $(\mathbb{C} - \{0\}, \times)$ has nontrivial elements of finite order such as $i = \sqrt{-1}$. On the other hand the $(\mathbb{C}, +)$ has no elements of finite order except the identity. Therefore these two groups can not be isomorphic.

5. (16pts) If a cyclic subgroup C of G is normal in G, then show that every subgroup of C is normal in G.

We have $C = \langle a \rangle$ for some $a \in G$. Suppose that H is a subgroup of C. Being a subgroup of a cyclic group, H must be cyclic as well. We have $H = \langle a^\ell \rangle$ for some integer ℓ. We need to show that $gHg^{-1} \subseteq H$ for every $g \in G$. The normality of C in G implies that $gag^{-1} = a^k$ for some integer k. Pick some element $x \in H$. Then there exists an integer m such that $x = (a^\ell)^m = a^{\ell m}$. Using the fact that $gag^{-1} = a^k$, we obtain that $gxg^{-1} = ga^{\ell m}g^{-1} = a^{k\ell m} = (a^\ell)^{km} \in H$. This finishes the proof.

6. (10pts) Let $n \geq 3$ be an integer. Consider the map $f : S_n \to S_n$ defined by the formula $f(\sigma) = \sigma^2$. Show that f is <u>not</u> a homomorphism.

Pick $\sigma = (12)$ and $\tau = (13)$ in S_n. Note that this is possible since $n \geq 3$. We have $f(\sigma\tau) = f((132)) = (123)$. On the other hand $f(\sigma) = (1)$ and $f(\tau) = (1)$. It follows that $f(\sigma\tau) \neq f(\sigma)f(\tau)$ for some elements σ and τ in S_n. Therefore f is not homomorphism from S_n to S_n.