M E T U
Department of Mathematics

Group	Discrete Mathematics Midterm 2	List No.

Code : *Math 112* Acad. Year : *2010* Semester : *Spring* Instructor : *A. B., K. Z., Ö. K.* Date : *May 6, 2010* Time : *17:50* Duration : *90 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES 60 TOTAL POINTS

1	2	3	4	5	6		

- **Justify your answers!** Correct answers without explanation may receive no credit.
- **There are 6 questions on 4 pages.** Check for typographical errors before the exam.

1. (10pts) Find the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where x_i are integers such that $1 \le x_i \le 6$ for all $1 \le i \le 4$.

Substitute $y_i = x_i - 1$ to get

$y_1 + y_2 + y_3 + y_4 = 13$, $0 \le y_i \le 5$ for all $1 \le i \le 4$.

$S_0 = \binom{13+4-1}{13} = \binom{16}{13}$

Let c_i be the condition that $y_i \ge 6$.

Then $N(c_1)$ = nb of solns to $y_1 + y_2 + y_3 + y_4 = 7$ where $y_i \ge 0$, for $1 \le i \le 4$. So $N(c_1) = \binom{7+4-1}{7} = \binom{10}{7}$

$N(c_1 c_2)$ = nb of solns to $y_1 + y_2 + y_3 + y_4 = 1$ where $y_i \ge 0$ for $1 \le i \le 4$. So $N(c_1 c_2) = \binom{1+4-1}{1} = \binom{4}{1}$

$N(c_1 c_2 c_3) = 0$. Hence

$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = \binom{16}{13} - 4\binom{10}{7} + \binom{4}{2}\binom{4}{1}$.

2. **(10pts)** When we throw a die 12 times, in how many distinct ways can all 6 numbers appear?

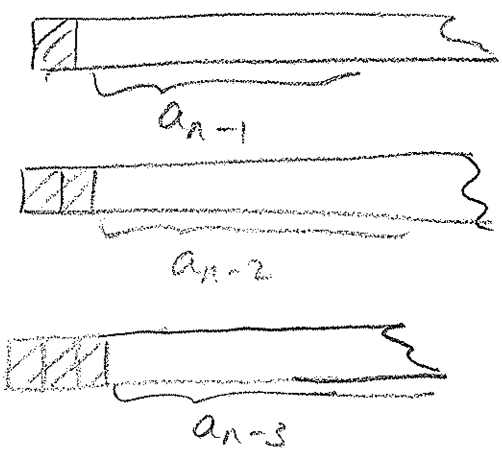
Let c_i be the condition that i does not appear, for $1 \le i \le 6$. Then $N(\bar{c}_1 \bar{c}_2 \cdots \bar{c}_6) = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 + S_6$

$S_0 = 6^{12}$, $S_1 = 6N(c_1) = 6 \cdot 5^{12}$, $S_2 = \binom{6}{2} N(c_1 c_2) = \binom{6}{2} 4^{12}$

$\dots, S_5 = 6, S_6 = 0$.

Ans: $6^{12} - 6 \cdot 5^{12} + \binom{6}{2} 4^{12} - \binom{6}{3} 3^{12} + \binom{6}{2} 2^{12} - 6$.

3. **(8pts)** In climbing a staircase, you can move up one, two or three steps at a time. Find a recurrence relation and initial conditions for the number of ways you can climb a staircase with n steps (*Do not solve the recurrence relation*).



For $n \geq 4$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

$a_1 = 1, \ a_2 = 2, \ a_3 = 4$

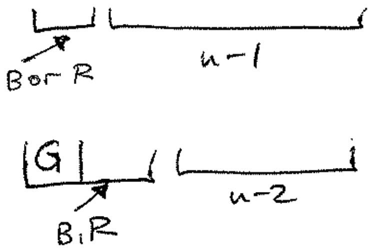
To climb 3 steps:

1+1+1 or 2+1 or 1+2 or 3

4. (8pts) Suppose we have an unlimited supply of red, blue and green cards. For $n \geq 1$, let a_n be the numbers of ways to arrange n cards in a line so that there is no consecutive green cards. Find a recurrence relation for a_n with initial conditions (*Do not solve the recurrence relation.*).

$a_1 = 3$

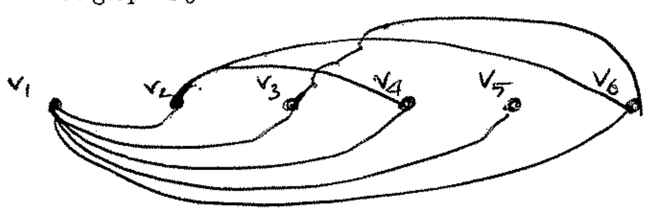
$a_2 = 3^2 - 1 = 8$



$a_n = 2a_{n-1} + 2a_{n-2}$ with initial conditions $a_1 = 3$, $a_2 = 8$

5. (12pts) Let $V = \{v_1, \ldots, v_k\}$ be the set of vertices in a loop-free undirected graph $G_k = (V, E)$. In this graph, two distinct vertices v_n and v_m are connected by an edge if and only if n divides m or m divides n.

a. Draw the graph G_6.



b. For which integer values $k \geq 1$, is G_k connected?

1 divides every positive integer $2 \leq n \leq k$. It follows that $\{v_1, v_n\} \in E$ for all $2 \leq n \leq k$. Given any two vertices v_n, v_m with $n \neq m$, there is a path between them $v_n - v_1 - v_m$ (WLOG $m, n \neq 1$). Therefore G_k is connected. for all $k \geq 1$.

c. For which integer values $k \geq 1$, is the complement $\overline{G_k}$ connected?

The vertex v_1 is isolated. in $\overline{G_k}$. Therefore $\overline{G_k}$ is disconnected for all $k \geq 2$. Note that $\overline{G_1}$ is trivially connected.

6. (12pts) Consider the recurrence relation

$$a_n - a_{n-1} - 6a_{n-2} = 3^n, \quad n \geq 2, \quad a_0 = 0, a_1 = 1.$$

a. Find $a_n^{(h)}$, the general solution to the **homogenous** equation

$$a_n - a_{n-1} - 6a_{n-2} = 0, \quad n \geq 2.$$

$$a_n^{(h)} = c \cdot r^n \Rightarrow c \cdot r^n - c \cdot r^{n-1} - 6cr^{n-2} = 0$$

$$\Rightarrow \underbrace{cr^{n-2}}_{\neq 0}(\underbrace{r^2 - r - 6}_{(r-3)(r+2)}) = 0$$

$$\Rightarrow r = -2, 3 \Rightarrow a_n^{(h)} = c_1(-2)^n + c_2(3)^n$$

b. Find a particular solution of the form $a_n^{(p)} = An3^n$.

$$An3^n - A(n-1)3^{n-1} - 6A(n-2)3^{n-2} = 3^{n-2}(\cancel{9An} - \cancel{3An} + 3A - \cancel{6An} + 12A)$$

$$= 3^n(\underbrace{\tfrac{5}{3}A}_{\text{should be } 1}) \Rightarrow A = \frac{3}{5}$$

c. Use parts **a** and **b** to solve the original **nonhomogenous** recurrence relation.

$$a_n = c_1(-2)^n + c_2(3)^n + \frac{3}{5} n \cdot 3^n$$

$$a_0 = 0 = c_1 + c_2$$

$$a_1 = 1 = -2c_1 + 3c_2 + \frac{9}{5}$$

$$\Rightarrow \begin{matrix} c_1 + c_2 = 0 \\ -2c_1 + 3c_2 = -\frac{4}{5} \end{matrix} \Rightarrow \begin{matrix} c_1 = \frac{4}{25} \\ c_2 = -\frac{4}{25} \end{matrix}$$

Therefore $a_n = \frac{4}{25}(-2)^n - \frac{4}{25}(3)^n + \frac{3}{5} n \cdot 3^n$